On the Utility Premium of Friedman and Savage [1]

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June 14, 2007

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The authors thank participants at the EGRIE conference in Barcelona as well as Arnold Chassagnon for helpful comments on an earlier draft of this paper.

Abstract

We re-examine the utility premium of Friedman and Savage (1948) and show how this somewhat neglected measure is actually quite useful in analyzing choice under risk. In particular, we show how the reaction of the utility premium to changes in wealth equates to a precautionary demand for saving. We also decompose the risk premium into two subcomponents: (1) the utility premium, which measures the degree of "pain" associated with a particular risk, and (2) a measure of willingness to pay to remove each unit of pain.

Keywords: Precautionary Saving, Prudence, Risk Aversion, Risk Premium, Utility Premium

JEL classification: D81

1 Introduction

A cornerstone for modern research on the economics of risk, at least in an expected-utility framework, has been the risk premium, which converts subjective preference towards risk into a monetary cost. To the best of our knowledge, the concept was originally introduced in a formal setting by Friedman and Savage (1948).[1] However, it was not until a seminal paper by Pratt (1964) introduced the risk premium in a more general setting that the measure became such an important tool for analyzing choice under risk. The fact that it can be linked to local properties of utility in an expected-utility framework was part of the beauty of Pratt. Using the risk premium as a starting point, many modern theories have examined how assumptions about risk attitudes can be captured via properties of the utility function.

This is in contrast to the utility premium, also introduced by Friedman and Savage (1948), which measures the degree of "pain" associated with risk, where pain is measured via a decrease in expected utility. Unlike the risk premium, the concept of the utility premium, which was exposited some sixteen years before Pratt's paper, has all but vanished from the literature on decision making under risk. This is likely due, in no small part, to the fact that the utility premium is not comparable

[1] Friedman and Savage (1948) introduced the notion of the "income equivalent" for a given risk, which in today's terminology is the "certainty equivalent." They also discuss the difference between the mean of a random wealth distribution and this income equivalent, which of course is the measure more formally defined by Pratt (1964) as the "risk premium."

between different individuals. However, behavior of the risk premium does allow us to analyze certain types of individual decisions under risk.

In this paper, we explain the relevance of the utility premium for decision making under risk. Whereas monotonicity of the risk premium in wealth, especially "decreasing absolute risk aversion," has an abundant literature exploring its implications for decisions in portfolio choice and insurance, we show how monotonicity of the utility premium in wealth has implications for two particular types of precautionary-savings models: one dealing with additive risks and the other with multiplicative risks.

2 Some properties of the utility premium

Let an individual's final wealth be represented by $w + \widetilde{\varepsilon}$, where $w > 0$ denotes the expected wealth of an individual and $\widetilde{\varepsilon}$ is a zero-mean random variable. The risk premium $\pi(w)$ for the risk $\widetilde{\varepsilon}$ at expected wealth w can be defined implicitly via

$$Eu(w + \widetilde{\varepsilon}) = u(w - \pi(w)), \tag{1}$$

where u denotes the individual's von Neumann-Morgenstern utility function and E denotes the expectation operator. We assume throughout this article that u is thrice differentiable with $u' > 0$ and $u'' < 0$. We also assume that the support of $\widetilde{\varepsilon}$ is defined such that $w + \varepsilon$ is in the domain of u.

Using "pain," as measured by a loss in utility from adding risk $\widetilde{\varepsilon}$, the utility premium can be defined as

$$v(w;\widetilde{\varepsilon}) \equiv u(w) - Eu(w + \widetilde{\varepsilon}). \tag{2}$$

Since we already know very much about Pratt's risk premium, we can use this knowledge to gain some insight into the utility premium. It is a tautology to write

$$\pi(w) = [u(w) - Eu(w + \widetilde{\varepsilon})] \times \left[\frac{\pi(w)}{u(w) - Eu(w + \widetilde{\varepsilon})}\right]. \tag{3}$$

The first term in (3) is simply the number of units of "pain" [utility premium] associated with risk $\widetilde{\varepsilon}$, whereas the second term is the average willingness to pay per unit of "pain" to eliminate the risk $\widetilde{\varepsilon}$. Since this is the only type of willingness to pay that we will consider, we denote is as simply "WTP."

2.1 An increase in wealth

The behavior of the risk premium as wealth changes has many well known implications. Ever since Arrow (1965), we have thought of decreasing absolute risk aversion (DARA) as a very canonical type of behavior: As one gets richer, one would not pay as much to remove the zero-mean risk $\widetilde{\varepsilon}$ from one's wealth. The decomposition in (3) allows us to see how the property of pain's decreasing in wealth is not the same property as DARA.

It follows from our definition that WTP must always be increasing in wealth under risk aversion. To see this, consider first that $d\pi/dw$ must always be strictly less than one. This follows trivially from (1), since $[1 - d\pi/dw] = Eu'(w+\widetilde{\varepsilon})/u'(w-\pi) > 0$. Thus,

$$\begin{aligned} \frac{dWTP}{dw} \times v^2(w;\widetilde{\varepsilon}) &= \left\{ v\frac{d\pi}{dw} - \pi[u'(w) - (1 - \frac{d\pi}{dw})u'(w-\pi)] \right\} \\ &> [1 - \frac{d\pi}{dw}]\pi[u'(w-\pi) - u'(w)] > 0. \end{aligned} \tag{4}$$

The first inequality in (4) follows under risk aversion, since we have $\pi u'(w) < v$. Hence, WTP is always increasing in wealth.

In one of the few papers that does examine the utility premium, Hanson and Menezes (1970) show that the utility premium is decreasing as one gets wealthier, i.e. $v'(w;\widetilde{\varepsilon}) \equiv \partial v(w;\widetilde{\varepsilon})/\partial w \leq 0$ for all w and all $\widetilde{\varepsilon}$, if and only if $u''' \geq 0$. This is easily seen to follow directly from (2) and Jensen's inequality. Of course it is well known that $u''' \geq 0$ is necessary but not sufficient for DARA. Still it interesting to consider a few examples:

Example 1 *Let utility be quadratic, $u(w) = w - kw^2$, $k > 0$, where we limit the domain of wealth to levels for which u is increasing. Since $u''' = 0$, as wealth increases, the level of pain from risk $\widetilde{\varepsilon}$ does not change. However, the willingness to pay to remove each unit of pain is increasing in wealth. Hence, as is well known, the risk premium is increasing in wealth.*

Example 2 *Let utility exhibit constant absolute risk aversion (CARA). Since $u''' > 0$, pain is decreasing wealth, whereas the WTP is increasing. Of course, under CARA, these two effects exactly offset one another. The amount one would pay to remove the entire risk $\widetilde{\varepsilon}$ remains constant as wealth changes.*

Example 3 *Under DARA, the level of pain is always decreasing as wealth increases, since $u''' > 0$, whereas the WTP to remove each unit of pain is always increasing in wealth, but at a slower rate.*

As we show in the next section, the property that pain is decreasing in wealth, whether or not we have DARA, has implications for precautionary models.[2]

[2]The above analysis is only for a fixed risk $\widetilde{\varepsilon}$, but can be easily extended to small risks by considering the random variable $t\widetilde{\varepsilon}$, as $t \to 0^+$. It is straightforward to show that WTP in this case converges to $1/u'(w)$. It also follows that "pain" for an infinitessimal risk is a second-order phenomenon, similar to the second order risk aversion of Segal and Spivak (1990). WTP, on the other hand, is a first-order effect.

2.1.1 Multiplicative Risks

The preceding analysis is based upon a fixed additive risk $\widetilde{\varepsilon}$. Much analysis in economics and finance deals with multiplicative risks. To this end, we now let wealth be defined as $w\widetilde{y}$ where $\widetilde{y} = 1 + \widetilde{\varepsilon}$. We maintain the assumption that $E\widetilde{\varepsilon} = 0$ and we assume that the support of $w\widetilde{y}$ is limited to levels of wealth over which preferences are well defined. The multiplicative risk premium $\widehat{\pi}(w)$ for a fixed random variable $\widetilde{y}$ is defined implicitly via

$$Eu(w\widetilde{y}) = u(w(1 - \widehat{\pi}(w))). \tag{5}$$

Note that $\widehat{\pi}(w)$ itself measures a proportion of wealth w, so that the monetary amount one is willing to forgo to eliminate the risk is equal to $w\widehat{\pi}(w)$.

We define the utility premium for the multiplicative risk $\widetilde{y}$ as

$$\widehat{v}(w; \widetilde{y}) = u(w) - Eu(w\widetilde{y}). \tag{6}$$

Hence, we can once again decompose the risk premium in to a measure of "pain" and a willingness to pay per unit of pain removed:

$$w\widehat{\pi}(w) = \widehat{v}(w; \widetilde{y}) \times \frac{w\widehat{\pi}(w)}{\widehat{v}(w; \widetilde{y})}. \tag{7}$$

Unlike in the additive case, where prudence alone was equivalent to the pain of an additive risk being decreasing in wealth, prudence no longer implies this result. We wish to evaluate the sign of $\widehat{v}'(w;\widetilde{y}) \equiv \partial\widehat{v}(w;\widetilde{y})/\partial w = u'(w) - E[\widetilde{y}u'(w\widetilde{y})]$. Once again applying Jensen's inequality, we see that $\widehat{v}'(w;\widetilde{y}) < [>]\ 0$ for all w and $\widetilde{y}$, if and only if the function $f(y) \equiv yu'(wy)$ is convex [concave]. Straightforward calculations show that f is convex [concave] if and only if relative prudence, $-wu'''(w)/u''(w)$, exceeds [is less than] two.[3] As an example, consider the very common assumption of constant relative risk aversion (CRRA).

Example 4 *Let utility exhibit CRRA with level of risk aversion $\gamma > 0$, so that either $u(w) \equiv \frac{1}{1-\gamma}w^{1-\gamma}$ for $\gamma \neq 1$, or $u(w) \equiv \ln w$. It follows that relative prudence is also constant, with $-wu'''(w)/u''(w) = 1 + \gamma$. Thus pain is decreasing [increasing] in wealth iff $\gamma > 1 \quad [\gamma < 1]$. Obviously, since relative risk aversion is constant, it follows from (7) that we must have WTP changing in the opposite direction from the risk premium.*

[3] We can consider the case of small risks by letting $\widetilde{y} \equiv 1 + t\widetilde{\varepsilon}$ and examining what happens as $t \to 0^+$. Once again, the level of pain is a second-order effect and the WTP is a first-order effect. It can be further shown that $[u'(w(1-\widehat{\pi}))]^{-1} <WTP< [u'(w)]^{-1}$, so that the average willingness to pay to remove a unit of pain is inversely proportional to marginal utility when the risk is infinitesimal.

3 Relation to Precautionary Saving

It is interesting to relate how changes in the utility premium with respect to wealth determine whether there is a precautionary savings demand in a two-period consumption-savings model. To this end, we consider two different models of precautionary saving, one in which the risk is additive and the other with the risk multiplicative.

Assume a two period model where both the risk-free interest rate and the rate of discounting for delayed consumption are both zero. The first model examines a consumer who has an income of w in the first period and an expected income of w in the second period. He or she decides on how much to save and how much to consume in the first period:

$$\max_{s} H(s;w) \equiv u(w-s) + Eu(w+\widetilde{\varepsilon}+s). \tag{8}$$

We do not necessarily restrict $s \geq 0$. Under risk aversion, it is trivial to show that the objective function H is globally concave in s. If there is no uncertainty in the second period ($\widetilde{\varepsilon} \equiv 0$), it is easy to see that the optimal savings is $s^* = 0$. For the general case, since H is concave in s, it follows that there is a precautionary demand for saving, $s^* > 0$, if and only if $[u'(w+s) - Eu'(w+\widetilde{\varepsilon}+s)]|_{s=0} < 0$, i.e. if and only if the utility premium is decreasing in wealth, $v'(w;\widetilde{\varepsilon}) < 0$. From our analysis in the previous section, this holds for every w and $\widetilde{\varepsilon}$ if and only if the decision maker is prudent.[4]

In the second model, we assume that the individual has an initial wealth of $2w$, with no other income. The individual must decide how much to save and how much to consume in the first period; but now we assume that the interest rate for savings is stochastic. The optimization program is

$$\max_s J(s;w) \equiv u(2w-s) + Eu(s(1+\widetilde{\varepsilon})). \tag{9}$$

The objective function J is globally concave in s. For $\widetilde{\varepsilon} \equiv 0$, the optimal savings strategy is to save one-half of initial wealth, $s^* = w$. For the general case, there will be a precautionary demand for saving, $s^* > w$, if and only if $[Eu'(s) - Eu'(s(1+\widetilde{\varepsilon}))]|_{s=w} < 0$, i.e. if and only in the utility premium for multiplicative risk is decreasing in wealth, $\widehat{v}'(w;\widetilde{\varepsilon}) < 0$. From our analysis in the previous section, this holds for every w and $\widetilde{\varepsilon}$ if and only if relative prudence exceeds two.[5]

We see in both models that a demand for precautionary saving follows if and only if the level of "pain" from the risk $\widetilde{\varepsilon}$ is decreasing in wealth. For the risky

[4]The relevance of signing u''' was known from the earlier works of Leland (1968) and Sandmo (1970), and was reexamined by Kimball (1990).

[5]The first proof of this result that we are aware of is by Rothschild and Stiglitz (1971), who at the time of their proof could not express their result in terms of "prudence."

second-period income model (8), we consider the additive risk $\tilde{\varepsilon}$. For the case of a risk-savings-rate model (9), we consider a multiplicative risk $\tilde{\varepsilon}$. In each model, by shifting some wealth to the second period, we reduce the "pain" of the $\tilde{\varepsilon}$ risk.

4 Concluding Remarks

The behavior of the utility premium with respect to changes in wealth was shown to explain the existence of a precautionary motive for savings in models for which there was an additive or a multiplicative risk in the future period. Although the utility premium is not comparable between individuals, how it changes in response to changes in wealth is something that can be used to predict particular comparative statics with respect to precautionary saving.[6]

Of course, in an abundance of situations, we need to know whether risk aversion itself is decreasing, for which a decreasing utility premium is a necessary but not a sufficient condition. In contrast, a decreasing utility premium, for which decreasing risk aversion is a sufficient condition, is required in generating a precautionary savings demand.

It is interesting to see how the result of Friedman and Savage (1948) is relevant

[6]In another paper, Eeckhoudt and Schlesinger (2006), we show how the shape of the utility premium as a function of wealth also has implications for higher-order risk effects.

in analyzing precautionary behavior under risk. It also is a bit surprising that a reexamination of their results, such as the one presented here, was mostly overlooked by the literature for the past 60 years.

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