

# Networks and Clubs

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## Abstract

Modeling club structures as bipartite networks, we formulate the problem of club formation as a game of network formation and identify those club networks that are stable if players behave farsightedly in choosing their club memberships. Using the farsighted core as our stability notion, we show that if players' payoffs are single-peaked and players agree on the peak club size (i.e., players agree on the optimal club size) and if there sufficiently many clubs to allow for the partition of players into clubs of optimal size, then a necessary *and* sufficient condition for the farsighted core to be nonempty is that players who end up in smaller-than-optimal size clubs have no incentive to switch their memberships to already existing clubs of optimal size. In contrast, we show via an example that if there are too few clubs relative to the number of players, then the farsighted core may be empty. Contrary to prior results in the literature involving myopic behavior, our example shows that overcrowding and farsightedness lead to instability in club formation.

## 1 Introduction

The study of group formation and group activities in economies has had a long history in economics, going back at least to Tiebout (1956) and Buchanan (1965). Such groups may form for the purposes of provision of public goods, either 'local' or pure, for the purpose of mutual insurance, to keep prices high (cartels), or simply for the members of the group to enjoy each others company. We shall call such groups

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‘clubs.’ Here we offer a new approach to the study of clubs. In particular, modeling club structures as bipartite networks, we formulate the problem of club formation as a game of network formation and identify those club networks that are stable if players behave farsightedly in choosing their club memberships. Thus we bring together two strands of the literature: club theory<sup>1</sup> and the theory of social and economic networks initiated by Kirman (1983).

Unlike the random graph theoretic approach taken by Kirman (1983), here we follow an approach similar to that taken by Jackson and Wolinsky (1996) in their study of networks and focus exclusively on strategic considerations in club network formation. The basic setup of our model is closely related to the model of Konishi, Le Breton and Weber (1997). They examine, however, free mobility equilibrium of a local public goods economy (an assignment of players to clubs, locations, or jurisdictions that partitions the population and has the property that no individual can gain by either moving to any other existing club, or creating his own club). The partition derived from the players’ strategy choices is thus stable against unilateral deviations by individuals.<sup>2</sup>

In contrast to the prior literature on clubs (discussed further below), we allow strategic coalitional moves and permit players to be farsighted.<sup>3</sup> Using the farsighted core introduced in Page and Wooders (2004) as our stability notion, we show that if players’ payoffs are single-peaked and players agree on the club size at which payoffs peak (i.e., players agree on the optimal club size) and if there are sufficiently many clubs (i.e., sufficiently many club types or club locations) to allow for the partition of players into clubs of optimal size, then a necessary *and* sufficient condition for the farsighted core to be nonempty is that players who end up in smaller-than-optimal size clubs (i.e., the left-over players) have no incentive to switch their memberships to already existing clubs of optimal size.<sup>4</sup> We note that in this case, the outcome of farsighted behavior corresponds to outcomes of myopic behavior as in Arnold and Wooders (2005) and the set of outcomes in the farsighted core correspond to the ‘Nash club equilibrium outcomes’.

The coincidence of outcomes of farsighted behavior and myopic behavior does not extend to all cases, however. We demonstrate via an example that if there are too few clubs relative to the number of players so that on average clubs must be larger than optimal size, then the farsighted core may be empty. This emptiness problem is caused by the fact that farsighted players, unlike myopic players, might switch their club memberships to already overcrowded clubs, temporarily making

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<sup>1</sup>See Demange and Wooders (2005), Part II for surveys of club theory from several perspectives.

<sup>2</sup>In a similar set up, Conley and Konishi (2002) analyze *migration proof* equilibrium, which are stable only against *credible* deviations on the part of a coalition. A coalitional deviation to another jurisdiction is credible if no outsiders to the coalition will want to follow the deviators and, within the deviating group, no player can gain by a further deviation. Conley and Konishi consider only the case where the number of possible clubs is unconstraining.

<sup>3</sup>Our approach differs from the cooperative/price-taking approach in much of the literature on clubs (again see Part II of Demange and Wooders) in that coalitions behave strategically.

<sup>4</sup>Stated loosely, a club network is contained in the farsighted core if no group of agents has an incentive to alter their club memberships, taking into account club membership changes that might take place in the future.

themselves worse off, if in the end switching induces an out-migration that makes them better off. We note that the Arnold and Wooders club formation model players behave myopically in choosing their club memberships and will switch memberships if and only if switching makes them strictly better off next period. Thus, in their model, since players are assumed to be unwilling to make themselves temporarily worse off, even if doing so induces payoff improving future out migrations, fewer membership defections are possible. As a result, Arnold and Wooders are able to show that even when there are too few clubs, a club structure in which all clubs are of nearly equal size is immune to coalitional defections. Using their terminology, Arnold and Wooders are able to show that if players are myopic and if there is overcrowding, a Nash club equilibrium always exists. In contrast, our analysis suggests that in general overcrowding and farsightedness may lead to instability in club formation.

Another important difference between farsighted and myopic behavior is demonstrated by an example. With myopic behavior, if individuals are spread too thinly among clubs, those in the smallest clubs might move into the biggest clubs, leading to a situation where all nonempty clubs have more than the optimal number of members. In contrast, from the same initial situation, if individuals are far-sighted then clubs can gradually build up to the optimal size. To the best of our knowledge, the differences in equilibrium outcomes of club economies, depending on whether individuals are farsighted or myopic, has not previously been noted in the literature.

Our framework builds on a very basic model of club formation: individuals are homogeneous and have single-peaked preferences over club size. The model captures the idea that individuals are positively affected by the number of members who share their clubs but eventually congestion effects set in so that there is a most preferred club size, which we shall call ‘optimal.’<sup>5</sup> The model has a long history in the literature, going back to club economies with essentially homogeneous players modelled as games in characteristic function form (cf., Pauly 1972 and Shubik and Wooders 1982,1983) and continuing to the more recent literature such as Banerjee, Konishi and Sonmez (2001), Bogomolnaia and Jackson (2002), Diamantoudi and Xue (2003). Banerjee, Konishi and Sonmez (2001) and Bogomolnaia and Jackson (2002) assume single-peaked preferences. Banerjee et al. focus on the core in simple coalition formation games while Bogomolnaia and Jackson study various solution concepts such as Nash stability and individual stability. Diamantoudi and Xue study the farsighted stable set and the largest consistent set (Chwe 1994) in these games. These papers underscore the importance of understanding the basic club model.

Our paper adds a network structure to the basic model. In the special case where the set of players can be divided into optimal sized clubs, then outcomes of all solution concepts coincide; nonemptiness of the core of the cooperative game is well-known and the robustness of the stability of partitioning players into clubs of optimal size is reinforced by our results. In the other case, however, when there are ‘too few’ clubs, the results mentioned above for cooperative games do not apply. In particular,

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<sup>5</sup>Note that this club size may not be Pareto-optimal; it is instead the preferred club size of an individual. The optimal club size may be three, for example, but optimality may require that, in a five-person economy, all individuals are in one club.

how is the characteristic function (stating the worth of each possible coalition) to be defined? The standard definition from economic theory defines the worth of a coalition as the most that it can guarantee itself no matter what the complementary coalition does. In our network framework (as in Konishi, Le Breton and Weber 1997), a coalition cannot prevent others from joining – that is, there is free entry.<sup>6</sup> Using the standard definition of the worth function, in all circumstances treated in this paper, the worth of a coalition would be the minimum of what it could achieve if the coalition constituted the membership of one club and of what it could guarantee itself if all players in the game were in a club with the coalition. In brief, the ‘free entry’ of our club model is not compatible with such a definition of what a coalition can achieve. (This is not at all meant as a criticism of the cooperative game approach; it addresses important, but different, sorts of situations.) The same comments apply to hedonic games (with ordinal preferences over coalitions of membership). The differences between our network approach and the cooperative game approach also appear in the set of equilibrium outcomes. We will elaborate on this with some examples in the paper.

On a more abstract level, in each of the papers on cooperative games noted above, there are essentially two primitives, a set of alternatives for each coalition and a dominance relationship. In our approach there are four primitives – the feasible set of networks, the preferences of players, the rules of network formation and a dominance relation. In this paper, given the player population, the feasible set of networks is determined by the number of club locations, the preferences of players are single-peaked over club size, the rules of network formation are free entry, and the dominance relations discussed are indirect and direct dominance. We focus primarily on indirect dominance (i.e., farsighted dominance). The importance of how one might arrive at a core point in a cooperative game has been long recognized. In the literature on networks based on cooperative games (cf., Slikker and van den Nouweland 2001 or van den Nouweland 2005 for a survey). Our framework allows us to consider this question in a club context.

We shall proceed as follows. In Section 2, we introduce the notion of a club network and state the assumptions of our model. In Section 3, we define the farsighted dominance relation over the feasible set of club networks, and we define the farsighted path dominance relation. In Section 4, we define the abstract club network formation game with respect to the farsighted path dominance relation and we define the farsighted core of the club network formation game. Finally, in Section 4, we state our main result giving necessary and sufficient conditions for nonemptiness of the farsighted core for the case in which there are sufficiently many clubs.

## 2 Clubs Networks

We begin by introducing the notion of a club network. Using bipartite networks we are able to represent in a very compact and precise way the totality of any given club structure.

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<sup>6</sup>See also Demange (1994) and, for a survey, Demange (2005).

Let  $N$  be a finite set of players consisting of two or more players with typical element denoted by  $i$ , and let  $C$  be a finite set of club types - or alternatively, a set of club labels or club locations - with typical element denoted by  $c$ .

**Definition 1** (*Club Networks*)

A club network  $g$  is a nonempty subset of  $N \times C$  such that  $(i, c) \in g$  if and only if player  $i$  is a member of club  $c$ .

Given club network  $g$ ,

$$g(c) := \{i \in N : (i, c) \in g\}$$

(i.e., the section of  $g$  at  $c$ ) is the set of members of club  $c$  in network  $g \subseteq N \times C$ , while the set

$$g(i) := \{c \in C : (i, c) \in g\}$$

(i.e., the section of  $g$  at  $i$ ) is the set of clubs to which player  $i$  belongs in network  $g \subseteq N \times C$ .

**Example 1** To illustrate, suppose there are five players  $N = \{i_1, i_2, i_3, i_4, i_5\}$  and two clubs  $C = \{c_1, c_2\}$ . Further, suppose that  $c_1$  denotes the chess club while  $c_2$  denotes the fencing club. Club network  $g_0$  depicted in Figure 1 represents one possible club structure given  $N$  and  $C$ .

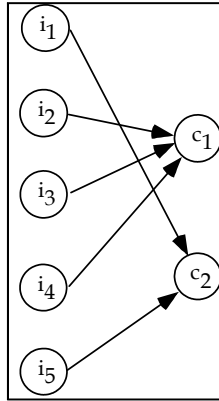


Figure 1: Club Network  $g_0$

In club network  $g_0$  the chess club has three members

$$g_0(c_1) = \{i_2, i_3, i_4\},$$

while the fencing club has two members

$$g_0(c_2) = \{i_1, i_5\}.$$

Note that in club network  $g_0$  each player is a member of one and only one club. Thus, for example

$$g_0(i_5) = \{c_2\},$$

that is, player  $i_5$  is a member of the fencing club, but is not a member of the chess club. Below we will formalize the single club membership property of this example in an assumption that we will maintain throughout the paper.

The collection of all club networks given  $N$  and  $C$  is given by the collection of all *nonempty* subsets of  $N \times C$ , denoted by  $P(N \times C)$ . We shall denote by  $|g(c)|$  the number of members of club  $c$  (i.e., the club size) in network  $g$  and by  $|g(i)|$  the number of clubs to which  $i$  belongs in network  $g$ . In Example 1, the chess club has three members, that is  $|g_0(c_1)| = 3$ , and player  $i_5$  belongs to one club - the fencing club - and thus  $|g_0(i_5)| = 1$ .

We shall maintain the following assumptions throughout:

**A-1** (single club membership) The feasible set of club networks,  $\mathbb{K} \subset P(N \times C)$ , is given by

$$\mathbb{K} \subseteq \{g \in P(N \times C) : |g(i)| = 1 \text{ for all } i \in N\}.$$

Thus, in each feasible club network  $g \in \mathbb{K}$  each player is a member of one and only one club. Again note that club network  $g_0$  in Example 1 satisfies the single club membership assumption [A-1]. Also note that under assumption [A-1] the collection  $\{g(c) : c \in C\}$  forms a partition of the set of players.

**A-2** (identical payoff functions depending on club size) Players have identical payoff functions,  $u(\cdot)$ , and payoffs are a function of club size only. In Example 1, player  $i_5$  is a member of the fencing club, that is,  $g_0(i_5) = \{c_2\}$ , and this club has a membership set given by

$$g_0(g_0(i_5)) := g_0^2(i_5) = \{i_1, i_5\}.$$

Thus, in network  $g_0$  player  $i_5$  has a payoff given by

$$u(|g_0(g_0(i_5))|) = u(|g_0^2(i_5)|) = u(|\{i_1, i_5\}|) = u(2).$$

In general, given any club network  $g$ ,  $|g^2(i)|$  denotes the total number of club members in the club to which player  $i$  belongs.

**A-3** (single-peaked payoffs) There exists a club size  $s^*$  with  $1 \leq s^* < |N|$  such that payoffs are increasing in club size up to club size  $s^*$  and decreasing thereafter.

**A-4** (free mobility) Each player can move freely and unilaterally from one club to another. This means that an player can drop his membership in any given club and join any other club without bargaining with or seeking the permission of any player or group of players. In this sense our model of club formation as a game over club networks is noncooperative. The assumption of free mobility is quite common in models of noncooperative network formation (see, for example, Bala and Goyal (2000)), as well as in the club literature (see, for example, Demange (2005) and the references contained therein).

**Example 2** *It is important to note that our assumptions do not rule out the possibility that some clubs have no members (i.e., are empty). Thus, in some feasible club networks  $g \in \mathbb{K}$ , it may be the case that  $g(c) = \emptyset$  for some club type  $c \in C$ . If club  $c$  has no members, then  $|g(c)| = |\emptyset| = 0$ . Figure 2 depicts just such a situation.*

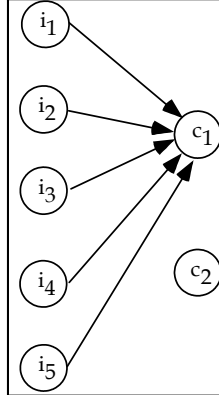


Figure 2: Club Network  $g_1$

*In moving from club network  $g_0$  in Example 1 to club network  $g_1$  above, players  $i_1$  and  $i_5$  have freely and unilaterally dropped their memberships in the fencing club and joined the chess club. Thus, in club network  $g_1$  the fencing club  $c_2$  has no members.<sup>7</sup>*

**Remark:** In the cooperative game/club context, which does not permit free entry, it would be standard to use  $u$  as the worth function of a cooperative game and then allow the game to be essentially superadditive – that is, a possibility open to a group of players is to divide into smaller clubs without any externalities between the clubs. Let us suppose that

$$\begin{aligned} u(1) &= 0, \\ u(2) &= 1, \\ u(3) &= 3 \\ u(n) &= 0, \quad n \geq 4. \end{aligned}$$

Suppose that there are 9 players in the game. When players can freely form coalitions (and exclude other players) the core is nonempty if and only if  $n$  is divisible by three. But we allow free entry into clubs, given in number, and it is here that we diverge from cooperative game theory.

Suppose still that there are 9 players. Also suppose that there are  $n \geq 1$  clubs. With free entry, if  $n \geq 4$ , the most that any coalition can *guarantee* itself is zero. This would give us the cooperative game  $(N, v)$  where  $v(S) = 0$  for all coalitions  $S$ .

<sup>7</sup>While we assume that in moving from club network  $g_0$  to club network  $g_1$  agents  $i_1$  and  $i_5$  act freely and unilaterally in switching their memberships, our model does not address the question of how agents  $i_1$  and  $i_5$  come to simultaneously switch their memberships, whether by communication and collusion or by serendipity. In order to formally address this question additional structure would have to be added to the current model. Page, Wooders, and Kamat (2005) make a start on addressing this question via the introduction of the supernetwork (i.e., a network of networks) in which the arcs represent coalitional moves and coalitional preferences (see also Page and Wooders (2004)).

The unique point in the core would be the zero vector. This is clearly not a fruitful approach.

One might try other notions of the core, for example, the gamma-core proposed by Chandler and Tulkens (1995, 1997). The gamma core assumes that the complementary coalition will do the best possible for itself. In some contexts, such as environmental economics, this is an eminently reasonable and fruitful approach. But it also does not give the same outcomes in all situations as our approach. Moreover, it does not allow us to consider farsightedness and consideration of the dynamics.

### 3 Dominance Relations Over Club Networks

Under the assumption of free mobility players can alter any existing club network by simply switching their memberships. Such membership changes however can trigger further membership changes by other players which in the end leave some or all of the players who initially switched not better off and possibly worse off. Here we will assume that players make their membership decisions taking into account the possibility of future membership changes by other players - that is, we will assume that players are farsighted and are concerned with the long run consequences of their immediate actions in choosing their club memberships. We begin by formalizing a notion of farsighted dominance. Then, using this farsighted dominance relation over club networks, we will identify club networks (i.e., club structures) that are farsightedly stable.

#### 3.1 Farsighted Dominance

Throughout let  $S$  denote a *nonempty* subset of  $N$ .

**Definition 2** (*Feasible Change and Improvement*) Let  $g_0$  and  $g_1$  be two club networks in  $\mathbb{K}$  ( $g_0 \neq g_1$ ).

(1) (Feasible Change) We say that players  $i \in S$  can feasibly change club network  $g_0$  to club network  $g_1$ , denoted

$$g_0 \xrightarrow{S} g_1,$$

if the move from network  $g_0$  to network  $g_1$  only involves a change in club memberships by players in  $S$ , leaving unchanged the memberships of players outside group  $S$ , that is, if

$$\text{if } g_0(i) = g_1(i) \text{ for all players } i \in N \setminus S \text{ (i.e., } i \text{ not contained in } S).$$

(2) (Improvement) We say that club network  $g_1$  is an improvement over club network  $g_0$  for players  $i \in S$ , denoted

$$g_1 \succ_S g_0,$$

$$\text{if } u(|g_1^2(i)|) > u(|g_0^2(i)|) \text{ for players } i \in S.$$

(3) (Feasible Improvement) We say that club network  $g_1$  is a feasible improvement over club network  $g_0$  for players  $i \in S$ , denoted

$$g_1 \triangleright_S g_0,$$

$$\text{if } g_0 \xrightarrow{S} g_1 \text{ and } g_1 \succ_S g_0.$$

(4) (Farsightedly Feasible Improvement) We say that club network  $g_* \in \mathbb{K}$  is a farsightedly feasible improvement over club network  $g \in \mathbb{K}$  (or equivalently, we say that club network  $g_*$  farsightedly dominates club network  $g$ ), denoted

$$g_* \triangleright \triangleright g,$$

if there exists a finite sequence of club networks,  $g_0, \dots, g_n$ , with  $g := g_0$  and  $g_* := g_n$ , and a corresponding sequence of sets of players,  $S_1, \dots, S_n$ , such that for  $k = 1, 2, \dots, n$ ,

$$g_{k-1} \xrightarrow{S_k} g_k \text{ and } g_k \succ_{S_k} g_{k-1}.$$

Thus, club network  $g_*$  is a farsighted feasible improvement over club network  $g$  if (i) there is a finite sequence of feasible changes in club networks starting with network  $g$  and ending with network  $g_*$ , and if (ii) payoffs

$$(u(g_*^2(i)))_{i \in N}$$

in ending club network  $g_*$  are such that for each  $k$  and for the players in each coalition  $S_k$ , payoffs in the ending club network  $g_*$  are greater than the payoffs players in  $S_k$  would have received in club network  $g_{k-1}$  (i.e., in the club network that players in  $S_k$  changed) - that is, for each  $k$

$$u(g_*^2(i)) := u(g_n^2(i)) > u(g_{k-1}^2(i)) \text{ for } i \in S_k.$$

The definition of farsighted feasible improvement above is a network rendition of Chwe's (1994) definition.

**Example 3** Suppose that there are seven players and two clubs and that the optimal club size is three. Figure 3 depicts three feasible club networks,  $g_0$ ,  $g_1$ , and  $g_2$ . Club

network  $g_2$  farsightedly dominates club network  $g_0$ .

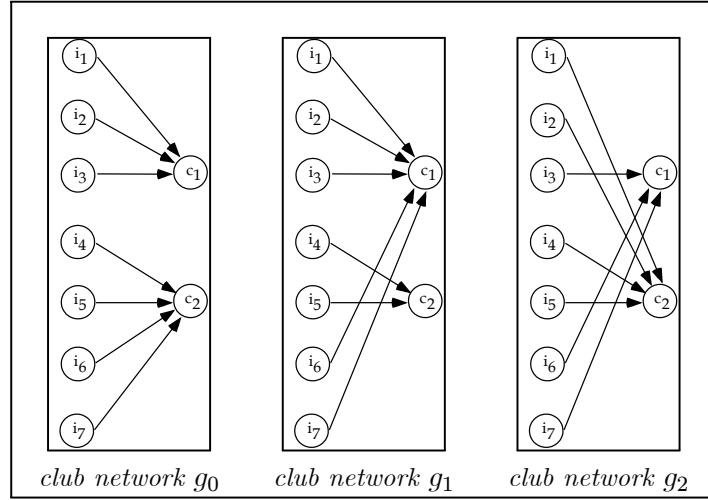


Figure 3: Three Possible Club Structures

To see this, consider the following sequence of moves. First, players  $i_6$  and  $i_7$  switch their memberships from club  $c_2$  to club  $c_1$ . This feasible move by players  $i_6$  and  $i_7$  changes club network  $g_0$  to club network  $g_1$  and is denoted by

$$g_0 \xrightarrow{\{i_6, i_7\}} g_1.$$

Second, players  $i_1$  and  $i_2$  switch their memberships from club  $c_1$  to club  $c_2$ . This feasible move by players  $i_1$  and  $i_2$  changes club network  $g_1$  to club network  $g_2$  and is denoted by

$$g_1 \xrightarrow{\{i_1, i_2\}} g_2.$$

Given an optimal club size of 3 and given the assumption of single-peaked payoffs, the initial moves by players  $i_6$  and  $i_7$  makes them worse off.<sup>8</sup> In particular, players  $i_6$  and  $i_7$  start out in club  $c_2$  in network  $g_0$  with 4 members  $\{i_4, i_5, i_6, i_7\}$  and payoffs given by

$$u(|g_0^2(i_6)|) = u(|g_0^2(i_7)|) = u(|\{i_4, i_5, i_6, i_7\}|) = u(4),$$

and move to club  $c_1$  creating a new club network  $g_1$  in which club  $c_1$  has 5 members  $\{i_1, i_2, i_3, i_6, i_7\}$ . As a result, players  $i_6$  and  $i_7$  are made worse off with payoffs given by

$$u(|g_1^2(i_6)|) = u(|g_1^2(i_7)|) = u(|\{i_1, i_2, i_3, i_6, i_7\}|) = u(5).$$

However, due to the second round of moves by players  $i_1$  and  $i_2$ , players  $i_6$  and  $i_7$  end up in a smaller club  $c_1$  in club network  $g_2$ , and thus end up better off. In particular, in the second round of moves, players  $i_1$  and  $i_2$  leave club  $c_1$  and move to club  $c_2$

<sup>8</sup>Allowing coalitions to initially be made worse off but then eventually better off, as in this example, differentiates farsighted dominance from other dominance relations.

- changing club network  $g_1$  to club network  $g_2$ . This move makes players  $i_1$  and  $i_2$  better off, but also makes players  $i_6$  and  $i_7$  better off. In particular, players  $i_1$  and  $i_2$  move from club  $c_1$  in network  $g_1$  with 5 members  $\{i_1, i_2, i_3, i_6, i_7\}$  and payoffs given by

$$u(|g_1^2(i_1)|) = u(|g_1^2(i_2)|) = u(|\{i_1, i_2, i_3, i_6, i_7\}|) = u(5),$$

to club  $c_2$  in network  $g_2$  with 4 members  $\{i_1, i_2, i_4, i_5\}$  and payoffs given by

$$u(|g_2^2(i_1)|) = u(|g_2^2(i_2)|) = u(|\{i_1, i_2, i_4, i_5\}|) = u(4).$$

These second round moves by players  $i_1$  and  $i_2$  leave players  $i_6$  and  $i_7$  in a smaller club  $c_1$  and thus make players  $i_6$  and  $i_7$  better off. Thus, players  $i_6$  and  $i_7$  who started out in club  $c_2$  in network  $g_0$  with 4 members  $\{i_4, i_5, i_6, i_7\}$  and payoffs given by

$$u(|g_0^2(i_6)|) = u(|g_0^2(i_7)|) = u(|\{i_4, i_5, i_6, i_7\}|) = u(4),$$

end up in club  $c_1$  in network  $g_2$  with 3 members,  $\{i_3, i_6, i_7\}$  and payoffs given by

$$u(|g_2^2(i_6)|) = u(|g_2^2(i_7)|) = u(|\{i_3, i_6, i_7\}|) = u(3).$$

### 3.2 Path Dominance

We say that a sequence of club networks  $\{g_k\}_k$  in  $\mathbb{K}$  forms a *farsighted domination path* (i.e., a  $\triangleleft\triangleleft$ -path) if for any two consecutive networks  $g_{k-1}$  and  $g_k$ ,

$$\begin{aligned} &g_k \text{ farsightedly dominates } g_{k-1}, \\ &\text{that is,} \\ &\text{if for any two consecutive networks } g_{k-1} \text{ and } g_k, \\ &g_{k-1} \triangleleft\triangleleft g_k. \end{aligned}$$

Using the terminology of graph theory, we can think of the farsighted dominance relation  $g_{k-1} \triangleleft\triangleleft g_k$  between club networks  $g_k$  and  $g_{k-1}$  as defining a  $\triangleleft\triangleleft$ -arc from network  $g_{k-1}$  to network  $g_k$ . The *length* of  $\triangleleft\triangleleft$ -path  $\{g_k\}_k$  is defined to be the number of  $\triangleleft\triangleleft$ -arcs in the path. We say that network  $g_1 \in \mathbb{K}$  is  $\triangleleft\triangleleft$ -reachable from network  $g_0 \in \mathbb{K}$  if there exists a *finite*  $\triangleleft\triangleleft$ -path in  $\mathbb{K}$  from  $g_0$  to  $g_1$ .

We can use the notion of  $\triangleleft\triangleleft$ -reachability to define a new relation on the feasible set of club networks  $\mathbb{K}$ . In particular, for any two networks  $g_0$  and  $g_1$  in  $\mathbb{K}$  define

$$g_1 \supseteq_{\mathbb{K}} g_0 \text{ if and only if } \begin{cases} g_1 \text{ is } \triangleleft\triangleleft \text{-reachable from } g_0 \text{ through } \mathbb{K}, \text{ or} \\ g_1 = g_0. \end{cases} \quad (1)$$

The relation  $\supseteq_{\mathbb{K}}$  is a weak ordering on  $\mathbb{K}$ . In particular,  $\supseteq_{\mathbb{K}}$  is reflexive ( $g \supseteq_{\mathbb{K}} g$ ) and  $\supseteq_{\mathbb{K}}$  is transitive ( $g_2 \supseteq_{\mathbb{K}} g_1$  and  $g_1 \supseteq_{\mathbb{K}} g_0$  implies that  $g_2 \supseteq_{\mathbb{K}} g_0$ ). We shall refer to the relation  $\supseteq_{\mathbb{K}}$  as the *farsighted path dominance relation (the FPD relation)*.<sup>9</sup>

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<sup>9</sup>The relation  $\supseteq_{\mathbb{K}}$  is sometimes referred to as the transitive closure in  $\mathbb{K}$  of the farsighted dominance relation,  $\triangleleft\triangleleft$ , on  $\mathbb{K}$ .

Note that if club network  $g_1$  is a feasible improvement over club network  $g_0$  for players  $i \in S$ , then  $g_1$  also dominates  $g_0$  with respect to the farsighted domination path (FDP) relation,  $\succeq_{\mathbb{K}}$ . Thus,

$$\text{if } g_1 \triangleright_S g_0 \text{ for some coalition } S, \text{ then } g_1 \succeq_{\mathbb{K}} g_0.$$

This applies even if the  $S$  consists of a single player, that is, even if  $S = \{i\}$  for some player  $i \in N$ . Thus,

$$\text{if } g_1 \triangleright_{\{i\}} g_0 \text{ for some player } i \in N, \text{ then } g_1 \succeq_{\mathbb{K}} g_0.$$

**Remark 1** *If network  $g_0 \in \mathbb{K}$  is  $\triangleleft\triangleleft$ -reachable from network  $g_0$ , then we say that  $\mathbb{K}$  contains a  $\triangleleft\triangleleft$ -circuit. Thus, a  $\triangleleft\triangleleft$ -circuit in  $\mathbb{K}$  starting at club network  $g_0 \in \mathbb{K}$  is a finite  $\triangleleft\triangleleft$ -path from  $g_0$  to  $g_0$ . A  $\triangleleft\triangleleft$ -circuit of length 1 is called a  $\triangleleft\triangleleft$ -loop. Note that because the relation  $\triangleleft\triangleleft$  is irreflexive (i.e., because it is not possible to have  $g \triangleleft\triangleleft g$ )  $\triangleleft\triangleleft$ -loops are in fact ruled out. However, because the farsighted dominance relation,  $\triangleleft\triangleleft$ , is not transitive, it is possible to have  $\triangleleft\triangleleft$ -circuits of length greater than 1.*

## 4 Club Formation Games and the Farsighted Core

A club formation game with farsighted players is a pair  $(\mathbb{K}, \succeq_{\mathbb{K}})$ , where  $\mathbb{K}$  is the feasible set of club networks and  $\succeq_{\mathbb{K}}$  is the farsighted domination path (FDP) relation on  $\mathbb{K}$ .

One of the most fundamental stability notions in game theory is the core. Here we define the notion of core for club formation games with respect to farsighted path dominance. We call this notion of the core the *farsighted core*.

**Definition 3** (*The Farsighted Core*)

*Let  $(\mathbb{K}, \succeq_{\mathbb{K}})$  be a farsighted club formation game. A subset  $\mathbb{C}$  of club networks in  $\mathbb{K}$  is said to be the farsighted core of  $(\mathbb{K}, \succeq_{\mathbb{K}})$  if for each club network  $g \in \mathbb{C}$  there does not exist a club network  $g' \in \mathbb{K}$ ,  $g' \neq g$ , such that  $g' \succeq_{\mathbb{K}} g$ .*

Note that any club network  $g$  contained in the farsighted core  $\mathbb{C}$  is a Nash club network - and in fact is a strong Nash club network.<sup>10</sup> Letting  $\mathbb{NE}$  denote the set of Nash club networks in  $\mathbb{K}$  and letting  $\mathbb{SNE}$  denote the set of strong Nash club networks in  $\mathbb{K}$ , we can conclude from our definition of the farsighted core that

$$\mathbb{C} \subseteq \mathbb{SNE} \subseteq \mathbb{NE}.$$

Example 3 is particularly interesting as it demonstrates that farsighted behavior may generate quite different outcomes than myopic behavior and strong Nash

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<sup>10</sup>A club network  $g \in \mathbb{K}$ , is a Nash club network if there does not exist another club network  $g' \in \mathbb{K}$  such that  $g' \triangleright_{\{i\}} g$  for some agent  $i \in N$ .

A club network  $g \in \mathbb{K}$ , is a strong Nash club network if there does not exist another club network  $g' \in \mathbb{K}$  such that  $g' \triangleright_S g$  for some coalition  $S$ .

equilibria (or Nash club equilibria). In Example 3, the number of clubs is not sufficiently large to permit all players to be in clubs of optimal size (i.e.,  $|C| < \frac{|N|}{s^*}$  for  $|C| = 2$ ,  $|N| = 7$ , and  $s^* = 3$ ). As shown in Arnold and Wooders (2002), in this case, it is a strong Nash equilibrium for the players to be divided into clubs that are as close as possible to the same size – in this example, into clubs of sizes 3 and 4. No group of players (nor any single player) can improve upon his own payoff – but, nevertheless, the farsighted core is empty. This is because, as the example illustrates, farsighted players, unlike myopic players, will switch their club memberships to an already overcrowded club, temporarily making themselves worse off, if in the end switching induces an out migration that makes them better off.

When the number of clubs is unconstraining, the situation is quite different. Our next results give necessary and sufficient conditions for the farsighted core of a club formation game to be nonempty when there is an ample number of clubs, that is, when the number of clubs is unconstraining.

**Theorem 1** (*Necessary and sufficient conditions for nonemptiness of the farsighted core*)

Consider a farsighted club formation game  $(\mathbb{K}, \succeq_{\mathbb{K}})$  with  $N$  players,  $C$  clubs, and optimal club size  $s^*$ ,  $1 \leq s^* < |N|$ . Suppose that assumptions (A-1)-(A-4) hold. In addition, assume that

- (a)  $|C| \geq \frac{|N|}{s^*}$ , and
- (b)  $|N| = rs^* + l$  for nonnegative integers  $r$  and  $l$ ,  $l < s^*$ .

The following statements are true.

1. The farsighted core of  $(\mathbb{K}, \succeq_{\mathbb{K}})$  is nonempty if and only if either  $l = 0$  or  $u(l) \geq u(s^* + 1)$ .
2. Club network  $g_*$  is contained in the farsighted core if and only if  $g_*$  has  $r$  clubs of size  $s^*$  and one club of size  $l$ .

**Proof.** Suppose that

$$|C| \geq \frac{|N|}{s^*} \text{ and } u(l) \geq u(s^* + 1).$$

Consider a club network  $g_*$  with  $r$  clubs of size  $s^*$  and one club of size  $l$  ( $l < s^*$ ). Let  $I$  be the group of players such that each player  $i$  in  $I$  is a member of an  $s^*$  club (i.e., a club of size  $s^*$ ) and let  $E$  be the group of players in the club of size  $l$ . Because

$$u(|g_*^2(i)|) \geq u(|g^2(i)|) \text{ for all } g \in \mathbb{K} \text{ and all } i \in I,$$

no coalition requiring the participation of players from  $I$  will be able to initiate a change in club network  $g_*$  which leads to another club network making the participants from  $I$  better off. Moreover, because

$$u(l) \geq u(s^* + 1) \text{ and payoffs are single peaked,}$$

no coalition of players from  $E$  alone will be able to initiate a change in club network  $g_*$  which leads to another club network making the players from  $E$  better off. Thus, for any club network  $g_*$  with  $r$  clubs of size  $s^*$  and one club of size  $l$ , there does not exist a club network  $g \in \mathbb{K}$ ,  $g \neq g_*$ , such that  $g \succeq_{\mathbb{K}} g_*$ . Therefore, if  $|C| \geq \frac{|N|}{s^*}$  and  $u(l) \geq u(s^* + 1)$ , then any club network  $g_*$  with  $r$  clubs of size  $s^*$  and one club of size  $l$  is in the farsighted core.

Suppose now that  $|C| \geq \frac{|N|}{s^*}$  but that  $u(l) < u(s^* + 1)$ . Let  $g \in \mathbb{K}$  and given  $g$  define the following club subcollections:

$$C_g^+ := \{c \in C : |g(c)| > s^*\},$$

$$C_g^* := \{c \in C : |g(c)| = s^*\},$$

and

$$C_g^- := \{c \in C : |g(c)| < s^*\}$$

Given that  $|C| \geq \frac{|N|}{s^*}$ ,  $C_g^- \neq \emptyset$  for all  $g \in \mathbb{K}$ .

Let  $g \in \mathbb{K}$  and suppose that  $C_g^+ \neq \emptyset$ . Consider clubs  $c_1 \in C_g^+$  and  $c_2 \in C_g^-$  and let  $S_1$  be a coalition of players from club  $c_1$  of size  $s^* - |g(c_2)|$ . Observe that if players in coalition  $S_1 \subseteq g(c_1)$  switch their memberships to club  $c_2$ , then the new larger club  $c_2$  will be of optimal size  $s^*$  and all members of coalition  $S_1$  will be made better off by making the switch. Let  $g' \in \mathbb{K}$  be the club network which results from this switch. Then we have

$$g' \succ_{S_1} g \text{ and thus } g' \succeq_{\mathbb{K}} g.$$

Let  $g \in \mathbb{K}$  and suppose that  $C_g^+ = \emptyset$ . If  $|C_g^*| = r$ , then there is a player  $i$  in some club  $c_1 \in C_g^-$  who can switch his membership to some club  $c_2 \in C_g^*$  and be made better off because  $u(l) < u(s^* + 1)$ . Letting  $g' \in \mathbb{K}$  be the club network resulting from this switch we have

$$g' \succ_{\{i\}} g \text{ and thus } g' \succeq_{\mathbb{K}} g.$$

If  $|C_g^*| < r$  (maintaining the assumption that  $C_g^+ = \emptyset$ ) then sufficiently many players from clubs in  $C_g^-$  can switch their memberships to some club  $c' \in C_g^*$  resulting in a new, larger club  $c'$  of optimal size  $s^*$ . Moreover, all players making this membership switch will be better off. Letting  $S'$  denote the coalition of players making the switch and letting  $g' \in \mathbb{K}$  be the resulting club network we have

$$g' \succ_{S'} g \text{ and thus } g' \succeq_{\mathbb{K}} g.$$

■

## 5 Path Dominance Induced by Direct Dominance

Throughout we have assumed that the path dominance relation on the feasible set of club networks  $\mathbb{K}$  is defined with respect to the farsighted dominance relation (i.e., an indirect dominance relation). We could also define path dominance with respect to

a direct dominance relation. We say that club network  $g_1$  is directly dominates club network  $g_0$ , denoted  $g_1 \triangleright g_0$ , if for some coalition  $S$

$$g_0 \xrightarrow{S} g_1 \text{ and } g_1 \succ_S g_0.$$

A sequence of club networks  $\{g_k\}_k$  is a *direct domination path* (i.e., a  $\triangleleft$ -path) through  $\mathbb{K}$  if for any two consecutive networks  $g_{k-1}$  and  $g_k$ ,

$$\begin{aligned} &g_k \text{ directly dominates } g_{k-1}, \\ &\text{that is,} \\ &\text{if for any two consecutive networks } g_{k-1} \text{ and } g_k, \\ &g_{k-1} \triangleleft g_k. \end{aligned}$$

Finally, a club network  $g_1$  path dominates club network  $g_0$  with respect to direct dominance (i.e., directly path dominates) if  $g_1 = g_0$  or if there is a finite  $\triangleleft$ -path in  $\mathbb{K}$  from  $g_0$  to  $g_1$ . In a manner analogous to our definition of the farsighted core, we define the direct dominance core as a subset  $\mathbb{C}_{\triangleright}$  of club networks in  $\mathbb{K}$  such that for each  $g \in \mathbb{C}_{\triangleright}$  there does not exist a club network  $g' \in \mathbb{K}$ ,  $g' \neq g$ , such that  $g'$  directly path dominates  $g$ .

Throughout we have also assumed that the rules of network formation are free entry (any number of players can unilaterally change their club memberships - see assumption (A-4)). Depending on the rules of network formation, direct path dominance and farsighted path dominance can yield very different results. For example, suppose we assume that the rules of network formation are noncooperative and free entry, that is, only one player at a time can unilaterally change his club membership. Also, suppose that there are 12 players and 6 club locations:

$$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

$$C = \{c_1, c_2, c_3, c_4, c_5, c_6\}.$$

Finally, suppose the optimal club size is 3 and that all players' single-peaked preferences over club sizes are given by

$$u(3) > u(2) > u(1)$$

$$\dots u(5) < u(4) < u(3)$$

$$u(4) > u(2) > u(1).$$

The club configuration

$\{1, 2, 3, 4\}$	$\{5, 6, 7, 8\}$	$\{9, 10, 11, 12\}$	$\emptyset$	$\emptyset$	$\emptyset$
$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$

with corresponding club network denoted by  $g_0$  is in the direct path dominance core under noncooperative, free entry rules, but is not in the farsighted core under

noncooperative, free entry rules. To see this consider the following sequence of non-cooperative moves. First, player 1 moves from club location  $c_1$  to club location  $c_4$ . After this move by player 1 the club configuration is

$$\boxed{\begin{array}{cccccc} \{2, 3, 4\} & \{5, 6, 7, 8\} & \{9, 10, 11, 12\} & \{1\} & \emptyset & \emptyset \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array}}$$

Let  $g_1$  be corresponding club network. Second, player 5 moves from club location  $c_2$  to club location  $c_4$ . After this move by player 5 the club configuration is

$$\boxed{\begin{array}{cccccc} \{2, 3, 4\} & \{6, 7, 8\} & \{9, 10, 11, 12\} & \{1, 5\} & \emptyset & \emptyset \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array}}$$

Let  $g_2$  be corresponding club network. Third, player 9 moves from club location  $c_3$  to club location  $c_4$ . After this move by player 9 the club configuration is

$$\boxed{\begin{array}{cccccc} \{2, 3, 4\} & \{6, 7, 8\} & \{10, 11, 12\} & \{1, 5, 9\} & \emptyset & \emptyset \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array}}$$

Let  $g_3$  be corresponding club network. Club network  $g_3$  is not only in the farsighted core, it is also in the direct path dominance core. To summarize the moves and preferences, we have

$$\begin{aligned} g_0 &\xrightarrow{\{1\}} g_1 \xrightarrow{\{5\}} g_2 \xrightarrow{\{9\}} g_3 \\ &\text{and} \\ g_0 &\prec_{\{1\}} g_3 \text{ and } g_1 \prec_{\{5\}} g_3 \text{ and } g_2 \prec_{\{9\}} g_3. \end{aligned}$$

Thus, club network  $g_3$  farsightedly dominates club network  $g_0$  - and thus, club network  $g_0$ , while being contained in the direct dominance core, is not contained in the farsighted core. This example illustrates the importance of the extent of farsightedness in whether optimal outcomes will be realized when individuals are initially spread too thin among clubs.

## 6 Conclusions

An aspect of our work which we find particularly interesting is relationships between the outcomes of the dynamic process in Arnold and Wooders (2002) and the outcomes of farsighted strategic behavior. Research in progress addresses these questions.

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