ELSEVIER

Journal of Mathematical Economics 39 (2003) 63–109

JOURNAL OF Mathematical ECONOMICS

www.elsevier.com/locate/jmateco

Three principles of competitive nonlinear pricing

Frank H. Page Jr. [a,*], Paulo K. Monteiro [b]

[a] *Department of Finance, University of Alabama, Tuscaloosa, AL 35487, USA*

[b] *EPGE/FGV Praia de Botafogo 190 sala 1103, 22253-900 Rio de Janeiro (RJ), Brazil*

Received 11 March 2002; received in revised form 14 June 2002; accepted 22 June 2002

Abstract

We make three contributions to the theory of contracting under asymmetric information. *First*, we establish a competitive analog to the revelation principle which we call the implementation principle. This principle provides a complete characterization of all incentive compatible, indirect contracting mechanisms in terms of contract catalogs (or menus), and allows us to conclude that in competitive contracting situations, firms in choosing their contracting strategies can restrict attention, without loss of generality, to contract catalogs. *Second*, we establish a competitive taxation principle. This principle, a refinement of the implementation principle, provides a complete characterization of all implementable nonlinear pricing schedules in terms of product–price catalogs and allows us to reduce any game played over nonlinear pricing schedules to a strategically equivalent game played over product–price catalogs. *Third*, applying the notion of payoff security (see Reny (1999)) and the competitive taxation principle, we demonstrate the existence of a Nash equilibrium for the mixed extension of the nonlinear pricing game. Moreover, we identify a large class of competitive nonlinear pricing games whose mixed extensions satisfy payoff security.

This paper extends earlier work by the first author (see Page (1992, 1999)).

Keywords: Competitive nonlinear pricing; Delegation principle; Implementation principle; Competitive taxation principle; Nash equilibria for discontinuous games

1. Introduction

For problems of screening and contracting under asymmetric information, the revelation principle states that given any incentive compatible, indirect contracting mechanism there exists an incentive compatible, direct contracting mechanism which in all circumstances

* Corresponding author. Tel.: +1-205-348-6097; fax: +1-205-348-0590.

E-mail addresses: fpage@cba.ua.edu (F.H. Page Jr.), pklm@fgv.br (P.K. Monteiro).

0304-4068/02/$ – see front matter © 2002 Elsevier Science B.V. All rights reserved.
PII: S0304-4068(02)00084-8

generates the same contract selections as the indirect mechanism (e.g. see Proposition 2, p. 73 in Myerson (1982)).[1] For screening problems in which a single firm (or principal) seeks to contract with several privately informed agents, the most important *implication* of the revelation principle is that in choosing a contracting strategy, the firm can restrict attention to direct contracting mechanisms without loss of generality. However, Martimort and Stole 1997) have shown via an example that for the polar opposite case of several firms competing to contract with a single privately informed agent, this important implication of the revelation principle no longer holds in general (see the example in Section 3.2 of Martimort and Stole (1997)). In particular, Martimort and Stole construct a two-firm, single-agent contracting game with a Nash equilibrium implementable via an indirect mechanism, but not implementable via any direct mechanism. Thus, it follows from the Martimort–Stole example that in multi-firm contracting games, restricting attention to direct mechanisms is no longer "without loss of generality". The economic intuition behind their example can be summarized as follows: because the range of a direct mechanism contains only those contracts which are chosen by some agent type, direct mechanisms can fail to take into account the possibility that in equilibrium some firms, in order to deter competing firms from defecting to other contracting strategies, might offer contracts which are not chosen by any agent type. The Martimort–Stole example thus highlights the fact that in competitive contracting games, contracts offered but not chosen can be critical to holding in place a particular equilibrium—or put differently, their example highlights the fact that in the case of competing firms, contracting strategies not only serve to resolve the adverse selection problem but also to deter defections. Because incentive compatible direct mechanisms address only the adverse selection problem, restricting attention to incentive compatible direct mechanisms is restrictive and not without loss of generality. Is there an analogue to the revelation principle for competitive contracting games?

This paper addresses this question and makes three contributions to the theory of competitive contracting under asymmetric information. First, within the context of a multi-firm contracting game, we provide a complete characterization of all incentive compatible, indirect mechanisms in terms of catalogs.[2] We call this result the *implementation principle*. This principle is analogous to the revelation principle in that it allows us to simplify multi-firm contracting games in a way similar to the way in which the classical revelation principle allows us to simplify single-firm, multi-agent contracting problems. The implementation principle states that given any profile of incentive compatible, indirect contracting mechanisms there exists a unique profile of contract catalogs which in all circumstances generates the same contract selections (by the agent) as the profile of indirect mechanisms; and *conversely*, that given any profile of contract catalogs, there exists a profile of indirect mechanisms which in all circumstances generates the same contract selections as the profile of catalogs.[3] Thus, it follows from the implementation principle that in competitive contracting situations,

[1] An indirect mechanism consists of a message space, M, and a function, $g(\cdot)$, from the message space into the set of contracts, K. If the agent sends message $m \in M$ to the principal, then contract $g(m) \in K$ is selected. A direct mechanism is simply a function, $f(\cdot) : T \to K$, defined directly on the set of agent types, T, taking values in the set of contracts, K. If the agent reports his type as $t' \in T$ to the principal, then contract $f(t') \in K$ is selected.

[2] A catalog is a set of contracts (see Section 3 for a formal definition and details).

[3] Here, since there are several firms (or principals), say m firms, we use the terminology profile to mean m-tuple.

firms in choosing their contracting strategies can restrict attention to catalogs without loss of generality. We shall refer to contracting games played over catalogs as catalog games.

Our second contribution is to establish a competitive analog to Guesnerie's taxation principle (see Guesnerie (1981, 1995) and Rochet (1985)). The *competitive taxation principle* we prove here states that given any profile of implementable nonlinear pricing schedules there exists a *unique* profile of product–price catalogs which in all circumstances generates the same product–price selections as the given profile of nonlinear pricing schedules; and *conversely*, that given any profile of product–price catalogs there exists a *unique* profile of implementable nonlinear pricing schedules which in all circumstances generates the same product–price selections as the given profile of catalogs.[4] Thus, by the competitive taxation principle, nonlinear pricing schedules and catalogs are strategically equivalent—and thus any noncooperative game played over nonlinear pricing schedules can be reformulated as a strategically equivalent noncooperative game played over product–price catalogs.

The competitive taxation principle can be viewed as a refinement of the implementation principle. While the implementation principle tells us that relative to the universe of all possible indirect mechanisms, no loss of generality is imposed by restricting attention to catalogs, the competitive taxation principle tells us that relative to the universe of implementable nonlinear pricing schedules (i.e. a particular class of indirect mechanisms), no loss of generality is imposed by restricting attention to product–price catalogs (i.e. a particular class of contract catalogs).[5] More importantly for applications, the competitive taxation principle shows us precisely how to extract from any given profile of product–price catalogs the corresponding unique profile of strategically equivalent nonlinear pricing schedules—and conversely, how to construct from any given profile of nonlinear pricing schedules the corresponding unique profile of strategically equivalent product–price catalogs.

Our third contribution is a Nash equilibrium existence result for catalog games. Two complications arise in proving existence. First, because the space of catalogs (a compact metric space) is not a vector space, the usual method of proving existence via a fixed point argument is not available. Thus, in order to address the existence question, we must introduce mixed strategies (or probabilistic strategies) over catalogs and consider the mixed extension of the catalog game. Second, except in the case of finitely many contracts, the mixed extension of the catalog game is oftentimes discontinuous.[6] Here, we are rescued by a recent result due to

[4] Let X be the set of all products (broadly defined) that firms can offer and let D (a subset of the nonnegative real numbers) be the set of all prices that firms can charge. A nonlinear pricing schedule for firm i is a pair $(X_i, p_i(\cdot))$ consisting of a product line, $X_i \subseteq X$ and a pricing function,

$$p_i(\cdot) : X_i \to D.$$

A product–price catalog for firm i is a subset C_i of $X \times D$.

[5] Thus, while the implementation principle guarantees that given any profile of contract catalogs, we can find in the universe of all possible indirect mechanisms a strategically equivalent profile of indirect mechanisms, the competitive taxation principle guarantees that given any profile of product–price catalogs, we can find in the universe of all possible implementable nonlinear pricing schedules a strategically equivalent profile of implementable nonlinear pricing schedules.

[6] In general, the agent's best response mapping is upper semicontinuous in catalog profiles (rather than continuous), and each firm's payoff function is upper semicontinuous in contracts (rather than continuous). The connection between nonexistence of Nash equilibria and lack of continuity in competitive contracting games was first illustrated via an example by Myerson (1982).

Reny (1999) on existence of Nash equilibria in discontinuous games. In particular, for mixed catalog games satisfying Reny's condition of payoff security, we are able to deduce, via Reny's existence result for discontinuous games, the existence of a Nash equilibrium.[7] Then, applying our existence result for mixed catalog games to mixed catalog games played over product–price catalogs, we are able deduce, via the competitive taxation principle, the existence of a Nash equilibrium for the mixed extension of the nonlinear pricing game. Are there catalog games whose mixed extensions satisfy payoff security? To answer this question, we introduce the notion of uniform payoff security, a condition implying payoff security. We then identify a large class of competitive nonlinear pricing games for which the corresponding catalog games satisfy uniform payoff security automatically. We conclude by showing that the mixed extension of any uniformly payoff secure catalog game is payoff secure.

A key ingredient in developing our understanding of the relationship between catalogs and indirect mechanisms is the *delegation principle* (see Page (1999)). This principle provides a complete characterization of all incentive compatible *and* individually rational direct contracting mechanism in terms of catalogs. Here, we shall refer to any such mechanism (i.e. any incentive compatible and individually rational direct contracting mechanism) as a *competitively viable* direct mechanism. Hammond (1979) proved the first delegation principle, characterizing incentive compatible, direct mechanisms in terms of catalogs for the single-principal, multi-agent problems, while Page (1999) established the delegation principle for the polar opposite case of several principals and one privately informed agent. Sharper results due to Page (1992) and Carlier (2000, 2001) characterize incentive compatible, direct mechanisms for single-principal, single-agent contracting problems.

In the competitive contracting environment considered here a direct contracting mechanisms is simply a mapping from agent types into firm–contract pairs, specifying for each possible agent type the firm and the contract the mechanism intends the agent to choose.[8] Under a competitively viable direct contracting mechanism the agent will participate in the mechanism and choose the firm–contract pair intended by the mechanism precisely because, by design, the mechanism provides no incentives for the agent to do otherwise. But in a competitive environment, who or what in the economy chooses such a mechanism? Such a mechanism would seem to require that firms choose a mediator who in turn chooses the mechanism—or at least, that firms act cooperatively in choosing the mechanism. This is precisely where the delegation principle comes into play. It follows from the delegation principle that the choice of a competitively viable direct contracting mechanisms can be *decentralized*. In particular, rather than have the agent report his type to a centralized contracting mechanism (however such a mechanism is chosen), instead each firm can simply offer the agent a catalog of contracts (i.e. a set of contracts *not* indexed by agent types) from which to choose.[9] Under this scenario, firms compete via their catalog offerings and agent choice is delegated. More importantly, under this scenario, firms can easily deduce the agent's best response mapping—a mapping which automatically incorporates all the

[7] Reny's notion of payoff security provides us with a precise way of identifying allowable discontinuities.

[8] Thus, here unlike Martimort and Stole (1997) we assume that the agent can contract with one and only one firm.

[9] This economic intuition first appears in the seminal paper by Hammond (1979) and underlies his characterization of direct, incentive compatible mechanisms (see Hammond (1979), p. 266, Theorem 1).

relevant market and agent specific information the agent might possess. We shall refer to this scenario as *anonymous delegated contracting*. Given a particular catalog profile offered by firms, any selection from the best response mapping (i.e. any function from agent types into firm–contract pairs summarizing the agent's optimal catalog choices) is a competitively viable direct mechanism. What is surprising is that the converse is also true: given any competitively viable direct mechanism there exists a unique, minimal catalog profile giving rise to the mechanism via the optimizing behavior of the agent under anonymous delegated contracting.[10] Stated more formally, by the delegation principle, a direct contracting mechanism is competitively viable if and only if there is a unique, minimal catalog profile implementing the mechanism.

The fact that competitively viable direct mechanisms can be decentralized via catalogs allows us to define the notion of a *Nash mechanism*. This notion, in turn, provides us with a general understanding of why the main implication of the classical revelation principle fails in competitive contracting situations.[11] We say that a competitively viable direct mechanism is Nash if the unique, minimal catalog profile implementing the mechanism is a Nash equilibrium for the corresponding catalog game. Conversely, if a Nash equilibrium catalog profile for a given catalog game is the unique, minimal catalog profile implementing some competitively viable direct mechanism, then this mechanism is Nash and we say that the Nash equilibrium catalog profile possesses a Nash mechanism. In general, if for a given catalog game a Nash equilibrium catalog profile fails to possess a Nash mechanism, then this Nash catalog profile cannot be implemented via any competitively viable direct mechanism. Such a failure implies that in the competitive contracting situation represented by the given catalog game attention cannot be restricted to competitively viable direct mechanisms without loss of generality (i.e. the main implication of the revelation principle fails). Here, we construct two examples of competitive contracting games illustrating the failure of the "without loss of generality" implication of the classical revelation principle. Unlike the Martimort and Stole (1997) example, in our example both the problems of adverse selection and strategic deterrence are present.[12] In our first example, the catalog game has a unique Nash equilibrium catalog profile with no corresponding Nash mechanism. In our second example, the catalog game has two Nash equilibria, one without a Nash mechanism and one with a Nash mechanism.

Despite the fact that the "without loss of generality" implication of the classical revelation principle fails to hold in competitive situations, it follows from the delegation and implementation principles that the revelation principle itself still holds in competitive situations. Here, we establish, as a corollary to the delegation and implementation principles, a *competitive revelation principle*. While the competitive revelation principle guarantees that the contract selections generated under any given incentive compatible profile of indirect contracting mechanisms can be replicated, agent type by agent type, by some competitively viable direct mechanisms, it does not guarantee—as confirmed by our examples—that the replicating

---

[10] The term *minimal* means smallest in terms of set inclusion (see Section 6.1 for a formal definition of minimal).

[11] The main implication being that "without loss of generality" attention can be restricted to competitively viable direct mechanisms.

[12] In the Martimort and Stole (1997) example the adverse selection problem is eliminated by assuming that the agent has only one type. Thus, in their example only the problem of strategic deterrence is present.

mechanism is Nash (for the unique, minimal catalog profile corresponding to the given profile of indirect contracting mechanisms). Thus, the competitive revelation principle *does not* restore the “without loss of generality” implication of the classical revelation principle.

2. Basic ingredients

2.1. Agent types and contracts

We shall assume that,

(A-1) The set of agent types is given by a probability space $(T, B(T), \mu)$, where T is a Borel space, $B(T)$ is the Borel σ-field in T, and μ is a probability measure defined on $B(T)$.

Recall that a Borel space is a Borel subset of a complete separable metric space. Under (A-1), multidimensional type descriptions are allowed.

Suppose now that there are m firms indexed by i and j $(=1, 2, \ldots, m)$ and let K be a set containing all possible contracts that firms can offer to the agent. Elements of K, denoted by f, can be viewed as describing the relevant characteristics of these contracts. For each firm $i = 1, 2, \ldots, m$, let K_i be a subset of K containing all the contracts that firm i can offer to the agent. The set K_i, then, is the ith firm’s feasible set of contracts.

We shall assume that,

(A-2) (i) K is a compact metric space containing an element 0 which we shall agree denotes “no contracting”, and (ii) for each firm $i = 1, 2, \ldots, m$, the feasible set of contracts K_i is a closed subset of K containing the element 0.

In order to take into account the possibility that the agent may wish to abstain from contracting altogether, we include in our list of feasible contract sets the set K_0 given by

$$K_0 := \{0\}. \tag{1}$$

Letting $I = \{0, 1, 2, \ldots, m\}$, define the set

$$\mathbb{K} := \{(i, f) \in I \times K : f \in K_i\}. \tag{2}$$

A pair $(i, f) \in \mathbb{K}$ indicates that the agent has chosen contract $f \in K_i$ from firm i, while $(0, 0) = (i, f) \in \mathbb{K}$ indicates that the agent has chosen to abstain from contracting altogether. Note that the set $\mathbb{K}$ is a closed subset of the compact metric space $I \times K$.[13] Thus, $\mathbb{K}$ is a compact metric space.

Examples (Contract sets).

[13] Equip I with the discrete metric $d_I^F(\cdot, \cdot)$ given by

$$d_I^F(i, i') = \begin{cases} 1, & \text{if } i \neq i', \\ 0, & \text{otherwise.} \end{cases}$$

(1) *Finite contract sets*: Perhaps the simplest example of a contract set K satisfying (A-2)(i) is the *finite contract set* given by a set of the form,

$$K := \{0, f^1, f^2, \dots, f^q\},$$

where 0 denotes "no contracting", and q is some positive integer greater than or equal to 1 denoting the number of contracts.

(2) *Products and prices*: Let X be a compact metric space representing the set of all products firms can offer and let D be a closed bounded subset of the real numbers representing the prices firms can charge. Assume that X contains a "no contracting" choice 0 and that D contains the zero price. The contract set

$$K := X \times D,$$

satisfies (A-2)(i), with $(0, 0) \in X \times D$ denoting "no contracting". Here, a contract is given by a product–price pair $(x, p) \in X \times D$. We shall return to this specification of the contract set K when we discuss nonlinear pricing.

(3) *Probabilistic contract sets*: Let Y be a compact metric space containing an element 0 denoting "no contracting" (for example, Y might be the set of contracts given by $X \times D$ in example (2) above) and let $\Delta(Y)$ denote the set of all probability measures defined on Y. Rather than take as the contract set the set Y, instead we might take as the contract set the set $\Delta(Y)$. Since Y is compact metric space, so too is $\Delta(Y)$ (see Aliprantis and Border (1999), Chapter 14). Moreover, $\Delta(Y)$ automatically contains an element representing "no contracting", namely the probability measure assigning probability 1 to the choice $0 \in Y$. Thus, the contract set

$$K := \Delta(Y),$$

satisfies (A-2)(i).[14] In this example a contract is given by a probability measure $f \in \Delta(Y)$.

(4) *State-contingent contracts*: Let $(Z, \Im, \eta)$ be a probability space where Z is the set of all possible states, $\Im$ is a σ-field of events in Z, and η is a probability measure. Consider the contract set K given by a set $\Im$-measurable functions defined on the state space Z taking values in some closed bounded interval $[L, H]$, where $L \leq 0 < H$.[15] In this example a contract is given by a function $f = f(\cdot) \in K$. Assume that the set of state-contingent contracts, K, has the following properties:

(i) K contains a function,

$$0(\cdot) : Z \to [L, H],$$

such that

$$0(z) = 0, \quad \text{for all } z \in Z.$$

[14] Thus, the model we construct here includes as a special case probabilistic contracts. Martimort and Stole (1997) focus on this particular case.

[15] See Section A.1 in the Appendix for the definition of $\Im$-measurability.

(ii) K is sequentially compact for the topology of pointwise convergence on Z, that is, for any sequence $\{f^n\}_n$ in K there is a subsequence $\{f^{n_k}\}_k$ in K and a function $\bar{f} \in K$ such that

$$f^{n_k}(z) \to \bar{f}(z), \quad \text{for all } z \in Z.$$

(iii) K contains no redundant contracts, that is, if f and $\bar{f}$ in K are such that

$$f(z') \neq \bar{f}(z'), \quad \text{for some } z' \in Z, \text{ then } \qquad \eta\{z \in Z : f(z) \neq \bar{f}(z)\} > 0.$$

The uniform boundedness of K (by L and H) together with conditions (ii) and (iii) imply that K is compact metrizable for the topology of pointwise convergence on Z (see Proposition 1 in Tulcea (1973)). In particular, under (ii), (iii), and uniform boundedness,

$$d_\eta(f, \bar{f}) := \int_Z |f(z) - \bar{f}(z)| \, \mathrm{d}\eta(z),$$

defines a metric on K which generates the topology of pointwise convergence[16] and makes K a compact metric space. Taking as the "no contracting" choice the state-contingent contract $0(\cdot)$, contract set K satisfies (A-2)(i).

2.2. The agent's utility function

Let

$$v(t, \cdot, \cdot) : \mathbb{K} \to R \tag{3}$$

be the utility function corresponding to a type $t \in T$ agent. We shall maintain the following assumptions throughout:

(A-3) The function $v(\cdot, \cdot, \cdot) : T \times \mathbb{K} \to R$ is such that (i) for each $t \in T$, $v(t, \cdot, \cdot)$ is continuous on $\mathbb{K}$, (ii) for each $(i, f) \in \mathbb{K}$, $v(\cdot, i, f)$ is $B(T)$-measurable,[17] and (iii) for each $t \in T$, $v(t, i, 0) < v(t, 0, 0)$ for all $i \in \{1, 2, \ldots, m\}$.

Note that we allow the agent's utility to depend not only on the contract but also on brand name (i.e. the name of the firm with which the agent contracts). However, by (A-3)(iii) if the agent is to derive any utility from a firm's brand name beyond the reservation level, $v(t, 0, 0)$, then the agent must enter into a contract with the firm. Also, note that allowing utility to depend on brand names *does not* rule out the possibility that some (or all) types of the agent are completely indifferent to brand names. Finally, note that if K is finite, then (A-3)(i) (continuity) is satisfied automatically.

[16] Thus, for any sequence $\{f^n\}_n \subseteq K$ and $\bar{f} \in K$, $d_\mu(f^n, \bar{f}) \to 0$ if and only if

$$f^n(z) \to \bar{f}(z), \quad \text{for all } z \in Z.$$

[17] See Section A.1 in the Appendix for the definition of $B(T)$-measurability.

2.3. The firm's profit function

The jth firm's profit is given by the function,

$$\pi_j(\cdot,\cdot,\cdot) : T \times \mathbb{K} \to R. \tag{4}$$

We shall maintain the following assumptions throughout:

(A-4) For $j = 1, 2, \ldots, m$, the profit function $\pi_j(\cdot,\cdot,\cdot)$ is such that (i) for each $t \in T$, $\pi_j(t,\cdot,\cdot)$ is upper semicontinuous on $\mathbb{K}$, and (ii) $\pi_j(\cdot,\cdot,\cdot)$ is $B(T) \times B(\mathbb{K})$-measurable and μ-integrably bounded (i.e. there exists a μ-integrable function $\xi_j(\cdot) : T \to R$ such that for t a.e. $[\mu]$, $|\pi_j(t, i, f)| \le \xi_j(t)$ for all $(i, f) \in \mathbb{K}$).[18]

Example (Profit functions and the utility function). Suppose, as in example (4) above, that the contract set K is given by a set of state-contingent contracts. Also, suppose that each firm $j = 1, 2, \ldots, m$ has conditional probability beliefs over the states given by $\beta_j(\cdot|\cdot)$, while the agent has conditional probability beliefs given by $\zeta(\cdot|\cdot)$. Assume that for each event $E \in \mathfrak{I}$, the functions

$$\beta_j(E|\cdot) : T \to [0, 1] \quad \text{and} \quad \zeta(E|\cdot) : T \to [0, 1]$$

are $B(T)$-measurable.

The utility function: let

$$u(\cdot,\cdot,\cdot) : T \times I \times [L, H] \to R$$

be such that (i) for each $t \in T$, $u(t,\cdot,\cdot)$ is continuous on $I \times [L, H]$, and (ii) for each $(i, c) \in I \times [L, H]$, $u(\cdot, i, c)$ is $B(T)$-measurable. Given utility function $u(\cdot,\cdot,\cdot)$, the type t agent's (expected) utility over firm–contract pairs is given by

$$v(t, i, f) := \int_Z u(t, i, f(z))\zeta(\mathrm{d}z|t).$$

Firm profit functions: For $j = 1, 2, \ldots, m$, let

$$r_j(\cdot,\cdot,\cdot) : T \times I \times [L, H] \to R$$

be such that (i) for each $t \in T$, $r_j(t,\cdot,\cdot)$ is upper semicontinuous on $I \times [L, H]$, (ii) for each $i \in I$, $r_j(\cdot, i,\cdot)$ is $B(T) \times B([L, H])$-measurable, and (iii) $r_j(\cdot,\cdot,\cdot)$ is μ-integrably bounded (i.e. there exists a μ-integrable function $g_j(\cdot) : T \to R$ such that for t a.e. $[\mu]$, $|r_j(t, i, c)| \le g_j(t)$ for all $(i, c) \in I \times [L, H]$).[19] Given profit function $r_j(\cdot,\cdot,\cdot)$, the jth firm's (expected) profit over firm–contract pairs is given by

$$\pi_j(t, i, f) := \int_Z r_j(t, i, f(z))\beta_j(\mathrm{d}z|t).$$

[18] Here, $B(\mathbb{K})$ denotes the Borel σ-field in the compact metric space $\mathbb{K}$. See Section A.1 in the Appendix for the definitions of upper semicontinuity and $B(T) \times B(\mathbb{K})$-measurability.

[19] $B([L, H])$ denotes the Borel σ-field in the interval $[L, H]$.

Specified in this way, the agent's utility function satisfies (A-3) and each firm's profit function satisfies (A-4).

3. Mechanisms

3.1. *Direct contracting mechanisms*

A direct contracting mechanism is a $(B(T), B(\mathbb{K}))$-measurable function

$$t \to (i(t), f(t)), \tag{5}$$

defined on the set of agent types taking values in $\mathbb{K}$.[20] Thus, a direct contracting mechanism is a function that specifies for each possible agent type an intended contracting arrangement for that agent type. For example, under direct contracting mechanism $(i(\cdot), f(\cdot))$, it is intended that a type t agent choose contract $f(t)$ from firm $i(t)$. Whether or not this happens depends upon the incentives provided by the mechanism.

Given direct contracting mechanism $(i(\cdot), f(\cdot))$, if the agent chooses what is intended for his type, then the jth firm's expected profit is given by

$$\Pi_j(i(\cdot), f(\cdot)) = \int_T \pi_j(t, i(t), f(t))\, \mathrm{d}\mu(t), \tag{6}$$

while a type t agent's utility is given by

$$v(t, i(t), f(t)). \tag{7}$$

Under direct contracting mechanism $(i(\cdot), f(\cdot))$ the agent will participate in the mechanism and choose what is intended for his type only if the mechanism does not provide incentives to do otherwise, that is, only if the mechanism is individually rational and incentive compatible.

3.2. *Competitively viable direct mechanisms*

A direct contracting mechanism $(i(\cdot), f(\cdot))$ is said to be *incentive compatible* if for all agent types t and t' in T it is true that

$$v(t, i(t), f(t)) \geq v(t, i(t'), f(t')). \tag{8}$$

Thus, a contracting mechanism $(i(\cdot), f(\cdot))$ is incentive compatible (IC) if for all agent types t, the mechanism does not provide incentives for the agent to enter into a contract $(i(t'), f(t'))$ not intended for his type.

A contracting mechanism $(i(\cdot), f(\cdot))$ is *individually rational* if for all agent types t in T it is true that

$$v(t, i(t), f(t)) \geq v(t, 0, 0). \tag{9}$$

Here, $v(t, 0, 0)$ is a type t agent's utility level if the agent abstains from contracting altogether (i.e. if the agent chooses $(0, 0) \in \mathbb{K}$). Thus, a mechanism $(i(\cdot), f(\cdot))$ is individually rational (IR) if it is rational for all agent types to participate in the mechanism.

[20] See Section A.1 in the Appendix for a definiton of $(B(T), B(\mathbb{K}))$-measurability.

We shall refer to any direct mechanism $(i(\cdot), f(\cdot))$ satisfying the incentive compatibility constraints (8) as well as the individual rationality constraints (9) as a *competitively viable direct* mechanism. Note that because each contract set K_i (for $i = 0, 1, \ldots, m$) contains a "no contracting" choice, endogenous participation by firms *and* the agent is automatically built into the notion of competitive viability.

4. Catalogs

4.1. Catalogs and catalog profiles

For each $i = 1, 2, \ldots, m$, let C_i be a nonempty, closed subset of K_i. We can think of the subset C_i as representing a *catalog* of contracts that the ith firm might offer to the agent. For $i = 0, 1, 2, \ldots, m$, let $P_f(K_i)$ denote the collection of all possible catalogs, that is, the collection of *all nonempty, closed subsets of* K_i.[21] Since K_i is a compact metric space, the collection of catalogs, $P_f(K_i)$, equipped with the Hausdorff metric h is automatically a compact metric space (see Section A.1 in the Appendix for the definition of the Hausdorff metric and a discussion).

If firms compete via catalogs, then their strategy choices can be summarized via a catalog profile,

$$(C_1, \ldots, C_m). \tag{10}$$

Here, the ith component of the m-tuple $(C_1, \ldots, C_m)$ is the catalog offered by the ith firm to the agent. Let

$$\boldsymbol{P} := P_f(K_1) \times \cdots \times P_f(K_m)$$

denote the space of all catalog profiles. If $\boldsymbol{P}$ is equipped with the metric $h_{\boldsymbol{P}}$ given by

$$h_{\boldsymbol{P}}((C_1, \ldots, C_m), (C'_1, \ldots, C'_m)) := \max\{h(C_i, C'_i) : i = 1, 2, \ldots, m\}, \tag{11}$$

then the space of catalog profiles $(\boldsymbol{P}, h_{\boldsymbol{P}})$ is a compact metric space.

4.2. The agent's problem under anonymous delegated contracting

Given $m + 1$-tuple of catalogs,

$$(C_0, C_1, \ldots, C_m),$$

where $C_0 = K_0 := \{0\}$, the agent's choice set is given by

$$\Gamma(C_0, C_1, \ldots, C_m) := \{(i, f) \in \mathbb{K} : f \in C_i\}. \tag{12}$$

Given choice $(i, f) \in \Gamma(C_0, C_1, \ldots, C_m)$, a type t agent's utility is given by

$$v(t, i, f).$$

[21] Note that since $K_0 = \{0\}$, $P_f(K_0)$ consists of a one nonempty, closed subset, namely the set $\{0\}$.

The agent's choice problem under anonymous delegated contracting is given by

$$\max\{v(t, i, f) : (i, f) \in \Gamma(C_0, C_1, \ldots, C_m)\}. \tag{13}$$

Note that because the agent can choose to abstain from contracting altogether (by choosing from C_0), *participation is endogenously determined*.

Under assumptions (A-1)–(A-3), for each t the agent's choice problem (13) has a solution. Let

$$v^*(t, C_1, \ldots, C_m) := \max\{v(t, i, f) : (i, f) \in \Gamma(C_0, C_1, \ldots, C_m)\} \tag{14}$$

and

$$\Phi(t, C_1, \ldots, C_m) := \{(i, f) \in \Gamma(C_0, C_1, \ldots, C_m) : v(t, i, f) = v^*(t, C_1, \ldots, C_m)\}. \tag{15}$$

The function

$$v^*(t, \cdot, \ldots, \cdot) : P_f(K_1) \times \cdots \times P_f(K_m) \to R$$

gives a type t agent's optimal level of utility as a function of the catalog profile offered by firms. Thus, $v^*(t, \cdot, \ldots, \cdot)$ expresses a type t agent's induced preferences over catalog profiles. The set-valued mapping

$$(C_1, \ldots, C_m) \to \Phi(t, C_1, \ldots, C_m)$$

is a type t agent's best response mapping. For each catalog profile

$$(C_1, \ldots, C_m) \in P_f(K_1) \times \cdots \times P_f(K_m),$$

$\Phi(t, C_1, \ldots, C_m)$ is a nonempty closed subset of $\mathbb{K}$

The following Proposition summarizes the continuity and measurability properties of the mappings, Γ and Φ, and the optimal utility function, v^*.

Proposition (Continuity and measurability properties). *Suppose assumptions* (A-1)–(A-3) *hold. Then the following statements are true*. (a) *The choice correspondence* $\Gamma(C_0, \cdot, \ldots, \cdot)$ *is* $h_{\boldsymbol{P}}$*-continuous on the space of catalog profiles* $\boldsymbol{P}$ (*i.e. is continuous with respect to the metric* $h_{\boldsymbol{P}}$ *given in* (11)), (b) *The function* $v^*(\cdot, \cdot, \ldots, \cdot)$ *is such that* $v^*(t, \cdot, \ldots, \cdot)$ *is* $h_{\boldsymbol{P}}$*-continuous on* $\boldsymbol{P}$ *for each* $t \in T$, *and* $v^*(\cdot, C_1, \ldots, C_m)$ *is* $B(T)$*-measurable on* T *for each* $(C_1, \ldots, C_m) \in \boldsymbol{P}$. (c) *For each* $t \in T$, $\Phi(t, \cdot, \ldots, \cdot)$ *is* $h_{\boldsymbol{P}}$*-upper semicontinuous on* $\boldsymbol{P}$ *and* $\Phi(\cdot, \cdot, \ldots, \cdot)$ *is* $B(T) \times B(\boldsymbol{P})$*-measurable on* $T \times \boldsymbol{P}$.[22]

[22] Here, $B(\boldsymbol{P})$ denotes the Borel σ-field in the compact metric space $(\boldsymbol{P}, h_{\boldsymbol{P}})$. Moreover,

$$B(\boldsymbol{P}) = B(P_f(K_1)) \times \cdots \times B(P_f(K_m)),$$

where $B(P_f(K_j))$ denotes the Borel σ-field in the compact metric space $(P_f(K_j), h)$ (see Aliprantis and Border (1999), p. 146, Theorem 4.43).

The proof of the Proposition above follows from Proposition 4.1 and 4.2 in Page (1992). It is easy to show that if K is a finite contract set, then for each $t \in T$, $\Phi(t, \cdot, \dots, \cdot)$ is $h_{\boldsymbol{P}}$-continuous on $\boldsymbol{P}$.

4.3. Expected potential profit under anonymous delegated contracting

For $t \in T$ and $(C_1, \dots, C_m) \in \boldsymbol{P}$, let

$$\pi_j^*(t, C_1, \dots, C_m) = \max\{\pi_j(t, i, f) : (i, f) \in \Phi(t, C_1, \dots, C_m)\}. \tag{16}$$

The quantity,

$$\pi_j^*(t, C_1, \dots, C_m),$$

is the maximum profit attainable by firm j given agent type t and catalog profile,

$$(C_1, \dots, C_m).$$

Thus, each firm uses as its measure of the potential profit in state

$$(t, C_1, \dots, C_m) \in T \times \boldsymbol{P},$$

the maximum level of profit attainable in this state (recall that for each agent type $t \in T$, the agent is indifferent over his possible choices from $\Phi(t, C_1, \dots, C_m)$). Given assumptions (A-4)(i) and (A-4)(ii) and given the upper semicontinuity and measurability properties of the best response mapping (see the Proposition in Section 4.2 above), it follows from Proposition 4.3 in Page (1992) that the potential catalog profit function defined in expression (16) is upper semicontinuous on $\boldsymbol{P}$ and $B(T) \times B(\boldsymbol{P})$-measurable on $T \times \boldsymbol{P}$ (see Page (1992), p. 275). Note that if the contract set K is finite, then for each agent type t, the function,

$$(C_1, \dots, C_m) \to \pi_j^*(t, C_1, \dots, C_m),$$

is automatically $h_{\boldsymbol{P}}$-continuous on the space of catalog profiles, $\boldsymbol{P}$.[23]

Given catalog profile $(C_1, \dots, C_m) \in \boldsymbol{P}$ the jth firm's expected potential catalog profit is given by

$$\Pi_j(C_1, \dots, C_m) = \int_T \pi_j^*(t, C_1, \dots, C_m)\, \mathrm{d}\mu(t). \tag{17}$$

It follows from Fatou's Lemma (see Aliprantis and Border (1999), p. 407), that each firm's expected potential catalog profit function,

$$(C_1, \dots, C_m) \to \Pi_j(C_1, \dots, C_m),$$

is $h_{\boldsymbol{P}}$-upper semicontinuous on $\boldsymbol{P}$. Moreover, if the contract set K is finite, then each firm's expected potential catalog profit function is $h_{\boldsymbol{P}}$-continuous on $\boldsymbol{P}$.

[23] This follows from the continuity of $\pi_j(t, \cdot, \cdot, \cdot)$ on the finite set $\mathbb{K}$ and from the $h_{\boldsymbol{P}}$-continuity of $\Phi(t, \cdot, \dots, \cdot)$ on the finite set $\boldsymbol{P}$.

5. Competitive contracting, catalog games, and Nash equilibrium

A catalog game is given by $(P_f(K_j), \Pi_j)_{j=1}^m$ where the set of catalogs, $P_f(K_j)$, is the jth firm's contracting strategy set and Π_j is the jth firm's expected potential catalog profit function, given in expression (17).

Definition (Nash equilibrium for catalog games). A catalog profile

$$(C_1^*, \dots, C_m^*) \in P_f(K_1) \times \cdots \times P_f(K_m)$$

is a Nash equilibrium for the catalog game $(P_f(K_j), \Pi_j)_{j=1}^m$ if for all $j = 1, 2, \dots, m$

$$\Pi_j(C_j^*, C_{-j}^*) \geq \Pi_j(C_j, C_{-j}^*), \quad \text{for all } C_j \in P_f(K_j).$$

Two questions now arise: first, in analyzing competitive contracting situations can we restrict attention, without loss of generality, to catalog games? Second, in general, do catalog games have Nash equilibria?

The answer to the first question is not obvious. Besides catalogs, firms have available other types of contracting strategies. These other types of strategies fall into two broad categories: (i) direct contracting mechanisms (discussed above) and (ii) indirect contracting mechanisms (discussed in Section 6.2 below). Thus, in order to answer the first question we must have a clear and precise understanding of how catalogs, direct mechanisms, and indirect mechanisms are related. The delegation and implementation principles, presented in Section 6, provide just such an understanding. In particular, the delegation principle provides a complete characterization of competitively viable direct mechanisms in terms of catalogs, while the implementation principle provides a complete characterization of incentive compatible indirect mechanisms in terms of catalogs. By the implementation principle we will be able to conclude that, in general, in competitive contracting situations attention can be restricted to catalog games without loss of generality. An important refinement of the implementation principle is the competitive taxation principle. This principle, also presented in the next section, establishes the strategic equivalence of nonlinear pricing schedules (a particular class of indirect mechanisms) and product–price catalogs (a particular class of catalogs).

The second question—the existence question for catalog games—is also difficult. It is difficult for two reasons: (i) in catalog games, strategy spaces (i.e. spaces of catalogs) are not vector spaces, and (ii) catalog games are often discontinuous. We are able to overcome the first difficulty by moving to mixed strategies and we are able to overcome the second difficulty by using a beautiful existence result due to Reny (1999).

6. Three principles

6.1. The delegation principle and Nash mechanisms

In this section, we state the *delegation principle*. This principle provides a complete characterization of competitively viable direct contracting mechanisms in terms of catalog profiles.[24] By the delegation principle, all competitively viable direct mechanisms can be decentralized via catalogs. This fact makes possible the introduction of the notion of a Nash mechanism, and this notion in turn facilitates a general understanding of why the "without loss of generality" implication of revelation principle fails in competitive situations. We begin with a formal statement of the delegation principle.

6.1.1. The delegation principle

Theorem 1 (The delegation principle). *Suppose assumptions* (A-1)–(A-3) *hold. Let*

$$(i(\cdot), f(\cdot)) : T \to \mathbb{K}$$

be a direct contracting mechanism. The following statements are equivalent:

1. *The direct contracting mechanism* $(i(\cdot), f(\cdot))$, *is competitively viable, that is,* $(i(\cdot), f(\cdot))$ *satisfies the incentive compatibility constraints* (8) *and the individual rationality constraints* (9).
2. *There exists a unique, minimal catalog profile* $(C_1, \ldots, C_m) \in \boldsymbol{P}$ *implementing* $(i(\cdot), f(\cdot))$, *that is, there exists a catalog profile* $(C_1, \ldots, C_m) \in \boldsymbol{P}$ *such that*
   (a) $(i(t), f(t)) \in \Phi(t, C_1, \ldots, C_m)$ *for all* $t \in T$, *and*
   (b) *for all other catalog profiles* $(C'_1, \ldots, C'_m) \in \boldsymbol{P}$, *if* $(i(t), f(t)) \in \Phi(t, C'_1, \ldots, C'_m)$ *for all* $t \in T$, *then for* $j = 1, \ldots, m$ *such that* $i(t) = j$ *for some* $t \in T$, $C_j \subseteq C'_j$.

According to the delegation principle, a direct mechanism, $(i(\cdot), f(\cdot))$, is competitively viable if and only if it is a (measurable) selection from the best response mapping, $t \to \Phi(t, C_1, \ldots, C_m)$, for some unique, minimal catalog profile $(C_1, \ldots, C_m) \in \boldsymbol{P}$, and therefore, if and only if

$$v(t, i(t), f(t)) = \max\{v(t, i, f) : (i, f) \in \Gamma(C_0, C_1, \ldots, C_m)\}, \quad \text{for all } t \in T,$$

for some unique, minimal catalog profile $(C_1, \ldots, C_m)$. By part 2(b) of the delegation principle, the unique, minimal catalog profile $(C_1, \ldots, C_m)$ implementing $(i(\cdot), f(\cdot))$ is minimal in the sense that under *any other* implementing profile $(C'_1, \ldots, C'_m)$, it must be true that $C_j \subseteq C'_j$ for all firms j attracting some participation under the mechanism $(i(\cdot), f(\cdot))$.[25]

We shall denote by

$$\sum(C_1, \ldots, C_m) \tag{18}$$

the set of all (measurable) selections from the best response mapping,

$$t \to \Phi(t, C_1, \ldots, C_m).$$

[24] The delegation principle is proved in Page (1999). For the convenience of the reader we include a proof in Section A.2 of the Appendix.

[25] If under mechanism $(i(\cdot), f(\cdot))$ firm j' attracts no participation (i.e. $i(t) \neq j'$ for all $t \in T$), then $C_{j'} = \{(0, 0)\}$, but nothing conclusive can be said about the relationship between the catalogs $C_{j'}$ and $C'_{j'}$ in the implementing profiles $(C_1, \ldots, C_m)$ and $(C'_1, \ldots, C'_m)$.

By the Kuratowski–Ryll–Nardzewski Selection Theorem (see Aliprantis and Border (1999), p. 567), for any catalog profile,

$$(C_1, \ldots, C_m) \in \boldsymbol{P},$$

the set of competitively viable mechanisms, $\sum(C_1, \ldots, C_m)$, is nonempty.

6.1.2. Nash mechanisms

By the delegation principle, any competitively viable direct mechanism $(i(\cdot), f(\cdot))$ can be decentralized via a unique, minimal catalog profile $(C_1, \ldots, C_m) \in \boldsymbol{P}$. This fact allows us to define the notion of a Nash mechanism.

Definition (Nash mechanisms). We say that a competitively viable direct mechanism $(i(\cdot), f(\cdot))$ is Nash if the corresponding unique, minimal catalog profile $(C_1, \ldots, C_m)$ implementing $(i(\cdot), f(\cdot))$ is a Nash equilibrium for the catalog game $(P_f(K_j), \Pi_j)_{j=1}^m$.

Thus, if a Nash catalog profile $(C_1, \ldots, C_m)$ for the catalog game $(P_f(K_j), \Pi_j)_{j=1}^m$ is such that $(C_1, \ldots, C_m)$ is the unique, minimal catalog profile implementing some competitively viable direct mechanism $(i(\cdot), f(\cdot))$, then $(i(\cdot), f(\cdot))$ is Nash. However, as the examples we construct in Section 7 will show, it is possible for a catalog game to possess a Nash equilibrium catalog profile with no corresponding Nash mechanism—and therefore, a Nash equilibrium catalog profile which cannot be implemented by any competitively viable direct mechanism. Such a Nash equilibrium catalog profile must naturally contain contracts—not chosen by any agent type—which serve only to deter defections by competitors. The fact that some catalog games have Nash equilibria with no corresponding Nash mechanisms means that in analyzing competitive contracting situations, attention cannot be restricted to competitively viable direct mechanisms without loss of generality.

6.2. The implementation principle

In competitive contracting situations, catalogs allow the firm to simultaneously address the problems of deterrence and adverse selection. The question that now arises is this: relative to the universe of all indirect mechanisms, is there any loss of generality caused by firms restricting attention to catalogs? The answer to this question is no—and is provided to us by the implementation principle.

6.2.1. Strategic competition via indirect contracting mechanisms

Let M denote the set of all possible messages the agent can send to firms. An *indirect contracting mechanism* is a pair,

$$(M_j, g_j(\cdot)),$$

where $M_j \subseteq M$ is the jth firm's message space and $g_j(\cdot) : M_j \rightarrow K_j$ is the jth firm's message function specifying for each message $m_j \in M_j$ sent by the agent to firm j, a contract selection $g_j(m_j)$ from the jth firm's feasible set of contracts K_j. If firms compete via indirect contracting mechanisms, then their strategy choices are given by a profile of indirect contracting mechanisms,

$$((M_1, g_1(\cdot)), \ldots, (M_m, g_m(\cdot))). \tag{19}$$

6.2.2. Indirect contracting mechanisms and agent choice

In order to take into account the possibility that the agent may wish to abstain from contracting altogether, we include in our list of indirect mechanisms the mechanism,

$$(M_0, g_0(\cdot)) \tag{20}$$

where

$$M_0 := \{m_0\} \quad \text{and} \quad g_0(m_0) := 0.$$

Given $m + 1$-tuple of indirect mechanisms,

$$((M_0, g_0(\cdot)), (M_1, g_1(\cdot)), \ldots, (M_m, g_m(\cdot))),$$

the agent's choice set is given by

$$\Psi(M_0, M_1, \ldots, M_m) := \{(i, m) \in I \times M : m \in M_i\}, \tag{21}$$

where $I = \{0, 1, 2, \ldots, m\}$.[26] Given choice $(i, m) \in \Psi(M_0, M_1, \ldots, M_m)$, a type t agent's utility is given by

$$v(t, i, g_i(m)).$$

6.2.3. The implementation principle

Extending the definition of incentive compatibility given in Rochet (1985)[27] to the competitive case, a profile of indirect contracting mechanisms,

$$((M_1, g_1(\cdot)), \ldots, (M_m, g_m(\cdot))),$$

is said to be incentive compatible if and only if there exists a function,

$$t \to (i(t), m(t)), \tag{22}$$

such that

$$\left.\begin{array}{l} (i)\ t \to (i(t), g_{i(t)}(m(t))) \text{ is } (B(T), B(\mathbb{K}))\text{-measurable}, \\ (ii) \text{ for all } t \in T,\ (i(t), m(t)) \in \Psi(M_0, M_1, \ldots, M_m), \text{ and} \\ (iii) \text{ for all } t \in T, \\ v(t, i(t), g_{i(t)}(m(t))) = \max\{v(t, i, g_i(m)) : (i, m) \in \Psi(M_0, M_1, \ldots, M_m)\}. \end{array}\right\} \tag{23}$$

We shall refer to any function $(i(\cdot), m(\cdot))$ satisfying conditions (i)–(iii) in expression (23) as a *message selection* from $((M_1, g_1(\cdot)), \ldots, (M_m, g_m(\cdot)))$ (see Section A.3.1 in the

[26] Recall that firms are indexed by i and j.

[27] See, in particular, expression (11) in Rochet (1985, p. 118).

Appendix for a discussion of conditions sufficient to guarantee the existence of message selections).

We now state our main result on the relationship between incentive compatible profiles of indirect contracting mechanisms and contract catalogs.

Theorem 2 (The implementation principle). *Suppose assumptions* (A-1)–(A-3) *hold. Then the following statements are true*:

1. *For each incentive compatible profile of indirect contracting mechanisms,*

   $$((M_1, g_1(\cdot)), \ldots, (M_m, g_m(\cdot))),$$

   there exists a unique, minimal catalog profile,

   $$(C_1, \ldots, C_m) \in \boldsymbol{P},$$

   such that for each message selection,

   $$(i'(\cdot), m'(\cdot)),$$

   from $((M_1, g_1(\cdot)), \ldots, (M_m, g_m(\cdot)))$, *there exists a competitively viable direct contracting mechanisms*

   $$(i(\cdot), f(\cdot)) \in \sum(C_1, \ldots, C_m),$$

   such that

   $$(i(t), f(t)) = (i'(t), g_{i'(t)}(m'(t)), \quad \text{for all } t \in T.$$

2. *For each profile of catalogs,*

   $$(C_1, \ldots, C_m) \in \boldsymbol{P},$$

   there exists an incentive compatible profile of indirect contracting mechanisms,

   $$((M_1, g_1(\cdot)), \ldots, (M_m, g_m(\cdot))),$$

   such that for each competitively viable direct contracting mechanism

   $$(i(\cdot), f(\cdot)) \in \sum(C_1, \ldots, C_m),$$

   there exists a message selection

   $$(i'(\cdot), m'(\cdot)),$$

   from $((M_1, g_1(\cdot)), \ldots, (M_m, g_m(\cdot)))$, *such that*

   $$(i(t), f(t)) = (i'(t), g_{i'(t)}(m'(t)), \quad \text{for all } t \in T.$$

By construction (see expression (49) in the proof of the implementation principle given in the Appendix), the unique, minimal catalog profile $(C_1, \ldots, C_m) \in \boldsymbol{P}$, corresponding to the indirect contracting mechanism,

$$((M_1, g_1(\cdot)), \ldots, (M_m, g_m(\cdot))),$$

is such that given *any* message selection $(i'(\cdot), m'(\cdot))$ from

$$((M_1, g_1(\cdot)), \dots, (M_m, g_m(\cdot))),$$
$$(i'(\cdot), g_{i'(\cdot)}(m'(\cdot))) \in \sum(C_1, \dots, C_m).$$

Thus, by construction, the unique, minimal catalog profile, $(C_1, \dots, C_m)$, corresponding to $((M_1, g_1(\cdot)), \dots, (M_m, g_m(\cdot)))$ is such that

$$\begin{aligned}&\text{for all competitively viable mechanisms, } (i(\cdot), f(\cdot)) \in \sum(C_1, \dots, C_m), \text{ and}\\ &\text{for all message selections, } (i'(\cdot), m'(\cdot)),\\ &v(t, i(t), f(t)) = v(t, i'(t), g_{i'(t)}(m'(t))), \quad \text{for all } t \in T.\end{aligned} \tag{24}$$

Moreover, even if we drop the measurability condition (i) in expression (23), and require only that message selections satisfy conditions (ii) and (iii), expression (24) remains valid.

6.2.4. A corollary: the competitive revelation principle

The implementation principle, together with the delegation principle, imply that the revelation principle holds in competitive contracting situations. In particular, we can state the following competitive revelation principle as a corollary to the delegation and implementation principles:

Corollary 1 (The competitive revelation principle). *Suppose assumptions* (A-1)–(A-3) *hold. Then the following statements are true*:

1. *Given any incentive compatible profile of indirect contracting mechanisms,*

   $$((M_1, g_1(\cdot)), \dots, (M_m, g_m(\cdot))),$$

   and any message selection,

   $$(i'(\cdot), m'(\cdot)),$$

   from $((M_1, g_1(\cdot)), \dots, (M_m, g_m(\cdot)))$, *there exists a competitively viable direct contracting mechanisms,*

   $$(i(\cdot), f(\cdot)),$$

   such that

   $$(i(t), f(t)) = (i'(t), g_{i'(t)}(m'(t))), \quad \textit{for all } t \in T.$$

2. *Given any competitively viable direct contracting mechanisms,*

   $$(i(\cdot), f(\cdot)),$$

   there exists an incentive compatible profile of indirect contracting mechanisms,

   $$((M_1, g_1(\cdot)), \dots, (M_m, g_m(\cdot))),$$

   and a message selection,

   $$(i'(\cdot), m'(\cdot)),$$

such that

$$(i(t), f(t)) = (i'(t), g_{i'(t)}(m'(t)), \quad \textit{for all } t \in T.$$

Part 1 of the competitive revelation principle guarantees that the contract selections generated by any message selection corresponding to any given incentive compatible profile of indirect contracting mechanisms can be replicated, agent type by agent type, by some competitively viable direct mechanisms. However, part 1 of the competitive revelation does not guarantee that the replicating mechanism is Nash for the unique, minimal catalog profile corresponding to the given profile of indirect contracting mechanisms. Thus, the competitive revelation principle does not imply that in modeling and analyzing competitive contracting situations attention can be restricted to competitively viable direct mechanisms without loss of generality.

The proof of part 1 of the competitive revelation principle follows immediately from part 1 of the implementation principle. For the proof of part 2 of the competitive revelation principle, consider the following:

Let

$$(C_1, \ldots, C_m) \in \boldsymbol{P}$$

be the unique, minimal catalog profile corresponding to the given competitively viable direct mechanism

$$(i(\cdot), f(\cdot)).$$

For each $j = 1, \ldots, m$, let the jth firm's indirect contracting mechanism be given by

$$(C_j, e_j(\cdot)),$$

where

$$e_j(\cdot) : C_j \to K_j$$

is the *identity* mapping, and C_j, the jth firm's message space, is given by the jth component of the unique, minimal catalog profile implementing $(i(\cdot), f(\cdot))$. To complete the proof, observe that the set of all message selections from the indirect mechanism

$$((C_1, e_1(\cdot)), \ldots, (C_m, e_m(\cdot)))$$

is equal to the set,

$$\sum(C_1, \ldots, C_m),$$

of all competitively viable direct mechanism corresponding to the catalog profile $(C_1, \ldots, C_m)$.

6.3. *The competitive taxation principle*

While the implementation principle tells us that relative to the universe of all possible indirect mechanisms, no loss of generality is imposed by restricting attention to catalogs, the competitive taxation principle tells us that relative to the universe of implementable nonlinear pricing schedules (i.e. a particular class of indirect mechanisms), there is a particular class of catalogs (i.e. product–price catalogs) strategically equivalent to the class of implementable nonlinear pricing mechanisms. More importantly for applications, the competitive taxation principle shows us precisely how to extract from any given profile of product–price catalogs the corresponding unique profile of strategically equivalent nonlinear pricing schedules—and conversely, how to construct from any given profile of implementable nonlinear pricing schedules the corresponding unique profile of strategically equivalent product–price catalogs.

6.3.1. Strategic competition via nonlinear pricing schedules

In order to analyze strategic competition via nonlinear pricing schedules we must explicitly introduce price. To begin, assume that the set of all possible contracts is given by

$$K := X \times D. \tag{25}$$

Elements of X can be viewed as describing the relevant characteristics of contracts, while elements of D specify price. Thus, for example, we might assume that X is a closed bounded subset of R^L where each vector $x = (x_1, \dots, x_L) \in X$ describes the characteristics of a particular sales contract such as quantity, quality, and location, while D is a closed bounded interval in R.

For each firm $i = 1, 2, \dots, m$, the feasible set of contracts, K_i, is a subset of $K := X \times D$. We shall assume that

(A-2)* (i) X is a compact metric space, containing an element 0 which we shall agree denotes "no contracting", (ii) D is a closed bounded subset of R containing 0, and (iii) for each firm $i = 1, 2, \dots, m$, K_i is a closed subset of $K := X \times D$ containing (0, 0).

Given our more detailed specification of K, the "no contracting" set, K_0, is now given by $K_0 := \{(0, 0)\}$, and the set $\mathbb{K}$ is given by

$$\mathbb{K} := \{(i, x, p) \in I \times X \times D : (x, p) \in K_i\},$$

where as before, $I = \{0, 1, 2, \dots, m\}$.

We shall strengthen our assumption (A-3) concerning the agent's utility function as follows:

(A-3)* The function $v(\cdot, \cdot, \cdot, \cdot) : T \times \mathbb{K} \to R$ is such that (i) for each $t \in T$, $v(t, \cdot, \cdot, \cdot)$ is continuous on $\mathbb{K}$, (ii) for each $(i, x, p) \in \mathbb{K}$, the function $v(\cdot, i, x, p)$ is $B(T)$-measurable, (iii) for each $t \in T$, $v(t, i, 0, 0) < v(t, 0, 0, 0)$ for all $i \in \{1, 2, \dots, m\}$, and (iv) for each $t \in T$ and each (i, x, p), (i, x, p') in $\mathbb{K}$ with $p' > p$, $v(t, i, x, p') < v(t, i, x, p)$.

Note that (A-3)* modifies (A-3) by adding the assumption that the agent's utility is *strictly decreasing in price*.

Example (A finite contract set). A useful example is provided by case in which the sets X and D are finite. For example, let

$$X = \{0, 1, 2, \ldots, Q\},$$

where Q is some large positive integer, and let

$$D = \{0, 0.01, 0.02, \ldots, P\},$$

where P is some large integer multiple of 0.01. Here, each $x \in X$ represents quantity in discrete units and each $p \in D$ represents price in dollars and cents. In this case, assumptions (A-2)*(i) and (A-3)*(i) are satisfied automatically.

Definition (Implementable nonlinear pricing schedules). An implementable nonlinear pricing schedule is a pair, $(X_j, p_j(\cdot))$, where X_j is a nonempty, closed subset of X representing the jth firm's product line, and $p_j(\cdot)$ is a real-valued, lower semicontinuous function,[28] defined on X_j taking values in the set of prices D such that

$$\text{graph}\{p_j(\cdot)\} := \{(x, p) \in X \times D : p = p_j(x)\} \subseteq K_j.$$

If firms compete via nonlinear pricing schedules, then their strategy choices are given by a profile of nonlinear pricing schedules,

$$((X_1, p_1(\cdot)), \ldots, (X_m, p_m(\cdot))). \tag{26}$$

6.3.2. Nonlinear pricing schedules and agent choice

In order to take into account the possibility that the agent may wish to abstain from contracting altogether, we include in our list of schedules the schedule,

$$(X_0, p_0(\cdot)) \tag{27}$$

where $X_0 := \{0\}$ and $p_0(0) := 0$.

Given $m + 1$-tuple of nonlinear pricing schedules,

$$((X_0, p_0(\cdot)), (X_1, p_1(\cdot)), \ldots, (X_m, p_m(\cdot))),$$

the agent's choice set is given by

$$\Lambda(X_0, X_1, \ldots, X_m) := \{(i, x) \in I \times X : x \in X_i\}, \tag{28}$$

where $I = \{0, 1, 2, \ldots, m\}$.[29] Given choice $(i, x) \in \Lambda(X_0, X_1, \ldots, X_m)$, a type t agent's utility is given by

$$v(t, i, x, p_i(x)).$$

[28] See Section A.1 in the Appendix for a definition of lower semicontinuity.

[29] Again, recall that firms are indexed by i and j.

6.3.3. The competitive taxation principle

We now state our main result on the relationship between catalogs and nonlinear pricing schedules.

Theorem 3 (The competitive taxation principle). *Suppose assumptions* (A-1), (A-2)*, *and* (A-3)* *hold. Then the following statements are true*:

1. *For each profile of catalogs,*

$$(C_1, \dots, C_m),$$

there exists a unique, minimal profile of implementable nonlinear pricing schedules,

$$((X_1, p_1(\cdot)), \dots, (X_m, p_m(\cdot))),$$

such that for all direct contracting mechanisms,

$$(i(\cdot), x(\cdot), p(\cdot)) \in \sum(C_1, \dots, C_m), \textit{ we have for all } t \in T,$$
$$(i(t), x(t), p(t)) = (i(t), x(t), p_{i(t)}(x(t))), \text{ and } x(t) \in X_{i(t)}.$$

2. *For each profile of implementable nonlinear pricing schedules,*

$$((X_1, p_1(\cdot)), \dots, (X_m, p_m(\cdot))),$$

there exists a unique, minimal catalog profile,

$$(C_1, \dots, C_m),$$

such that for all direct contracting mechanisms,

$$(i(\cdot), x(\cdot), p(\cdot)) \in \sum(C_1, \dots, C_m), \textit{ we have for all } t \in T, (i(t),$$
$$x(t), p(t)) = (i(t), x(t), p_{i(t)}(x(t))), \textit{ and } x(t) \in X_{i(t)}.$$

It follows from the competitive taxation principle *and* the delegation principle that given any competitively viable direct contracting mechanisms there is a unique profile of implementable nonlinear pricing schedules which implements the mechanism. Conversely, given any profile of implementable nonlinear pricing schedules,

$$((X_1, p_1(\cdot)), \dots, (X_m, p_m(\cdot))),$$

with corresponding unique, minimal catalog profile,

$$(C_1, \dots, C_m),$$

there exists a unique set of direct mechanisms, $\sum(C_1, \dots, C_m)$, such that each mechanism, $(i(\cdot), x(\cdot), p(\cdot)) \in \sum(C_1, \dots, C_m)$, is uniquely implemented by the profile of nonlinear pricing schedules, $((X_1, p_1(\cdot)), \dots, (X_m, p_m(\cdot)))$.

Part 1 of the competitive taxation principle is established in Page (1999).[30] Moreover, Page (1999) shows that given any catalog profile $(C_1, \ldots, C_m)$, the profile of nonlinear pricing schedules,

$$((X_1, p_1(\cdot)), \ldots, (X_m, p_m(\cdot))),$$

which uniquely implements the catalog profile must be such that for each j, the product line, X_j, is closed and the pricing function, $p_j(\cdot)$, lower semicontinuous. Thus for each firm, closedness of the product line, X_j, and lower semicontinuity of the pricing function, $p_j(\cdot)$, are necessary conditions for implementation.

7. Two examples

In this section, we present two examples of competitive contracting games illustrating the failure of the "without loss of generality" implication of the revelation principle in competitive situations. In our first example, we construct a catalog game with a unique Nash equilibrium catalog profile—but a profile which does not possess a corresponding Nash mechanism. Our second example is constructed by altering the catalog game in our first example to obtain a game with two Nash equilibrium catalog profiles—one with a Nash mechanism and one without.

7.1. Example 1

7.1.1. Basic ingredients

Suppose there are two contracts, f_A and f_B, so that, $K := \{0, f_A, f_B\}$, and assume that there are two firms with

$$K_i := \{0, f_A, f_B\}, \quad \text{for } i = 1, 2.$$

The space of catalog profiles is given by

$$\boldsymbol{P} = P_f(K) \times P_f(K),$$

where

$$P_f(K) = \{\{0\}, \{f_A\}, \{f_B\}, \{0, f_A\}, \{0, f_B\}, \{f_A, f_B\}, \{0, f_A, f_B\}\}.$$

Suppose also that there are three agent types,

$$T := \{t_1, t_2, t_3\},$$

having equal probabilities. Thus,

$$\mu(t_1) = \mu(t_2) = \mu(t_3) = \tfrac{1}{3}.$$

The agent's ranking of firm–contract pairs, given by the agent's utility function, is as follows:

[30] For the convenience of the reader we include a proof of part 1 of the competitive taxation principle in Section A.4 of the Appendix.

Utility ranking:

$$\left.\begin{array}{l} v(t_1,(2,f_B)) > v(t_1,(1,f_A)) > v(t_1,(1,f_B)) > v(t_1,(0,0)) > v(t_1,(2,f_A)),\\ v(t_2,(1,f_A)) > v(t_2,(2,f_B)) > v(t_2,(2,f_A)) > v(t_2,(0,0)) > v(t_2,(1,f_B)),\\ v(t_3,(2,f_A)) > v(t_3,(1,f_B)) > v(t_3,(2,f_B)) > v(t_3,(0,0)) > v(t_3,(1,f_A)). \end{array}\right\} \tag{29}$$

Recall that by (A-3)(iii), for all agent types t,

$$v(t,(1,0)) < v(t,(0,0)) \quad \text{and} \quad v(t,(2,0)) < v(t,(0,0)).$$

Thus, we need only consider the catalog game played over the collection of catalogs given by

$$\{\{0\},\{f_A\},\{f_B\},\{f_A,f_B\}\}.$$

Profit functions are specified as follows:
Firm 1's profit:

$$\left.\begin{array}{l} \text{For all } t \in T,\\ \pi_1(t,(1,f_A)) = 6 \quad \text{and} \quad \pi_1(t,(1,f_B)) = 3,\\ \text{while}\\ \pi_1(t,(2,f_A)) = \pi_1(t,(2,f_B)) = 0,\\ \text{and}\\ \pi_1(t,(0,0)) = \pi_1(t,(1,0)) = \pi_1(t,(2,0)) = 0. \end{array}\right\} \tag{30}$$

Firm 2's profit:

$$\left.\begin{array}{l} \text{For all } t \in T,\\ \pi_2(t,(2,f_B)) = 6 \quad \text{and} \quad \pi_2(t,(2,f_A)) = 3,\\ \text{while}\\ \pi_2(t,(1,f_A)) = \pi_2(t,(1,f_B)) = 0,\\ \text{and}\\ \pi_2(t,(0,0)) = \pi_2(t,(1,0)) = \pi_2(t,(2,0)) = 0. \end{array}\right\} \tag{31}$$

Thus, firm 1 makes more profit selling contract f_A, while firm 2 makes more profit selling contract f_B. Each firm makes zero profit if the agent chooses a contract from the other firm.

7.1.2. Best responses and expected profits

With these basic ingredients in hand we can compute the agent's type-dependent best responses to each catalog profile offered by firms, and with these best responses (the selections from the best response mapping), we can compute each firm's expected profit given the prevailing catalog profile. Using these expected profit numbers, we can then construct the payoff matrix corresponding to catalog game. Here, we will carry out only some sample calculations to indicate how the entries in the payoff matrix are obtained.

Suppose firm 1 offers catalog

$$C_1 = \{f_A, f_B\},$$

and firm 2 offers catalog

$$C_2 = \{f_A, f_B\}.$$

Using the utility ordering given in expression (29), we can easily compute the agent's best responses to catalog profile, $(C_1, C_2) = (\{f_A, f_B\}, \{f_A, f_B\})$. We obtain

$$\begin{aligned}
&\Phi(t_1, \{f_A, f_B\}, \{f_A, f_B\}) = \{(2, f_B)\},\\
&\Phi(t_2, \{f_A, f_B\}, \{f_A, f_B\}) = \{(1, f_A)\},\\
&\Phi(t_3, \{f_A, f_B\}, \{f_A, f_B\}) = \{(2, f_A)\}.
\end{aligned}$$

Note that the best response mapping

$$t \to \Phi(t, \{f_A, f_B\}, \{f_A, f_B\})$$

is single-valued. Thus, there is only one (measurable) selection from the best response mapping (i.e. there is only one competitively viable direct mechanism corresponding to catalog profile $(\{f_A, f_B\}, \{f_A, f_B\})$), and it is given by

$$(i(t), f(t)) = \begin{cases} (2, f_B) \text{ if } t = t_1, \\ (1, f_A) \text{ if } t = t_2, \\ (2, f_A) \text{ if } t = t_3. \end{cases}$$

Given catalog profile $(\{f_A, f_B\}, \{f_A, f_B\})$, firm 1's expected profit is given by

$$\begin{aligned}
&\Pi_1(\{f_A, f_B\}, \{f_A, f_B\})\\
&\quad = \pi_1(t_1, (2, f_B))\mu(t_1) + \pi_1(t_2, (1, f_A))\mu(t_2) + \pi_1(t_3, (2, f_A))\mu(t_3),
\end{aligned}$$

while firm 2's expected profit is given by

$$\begin{aligned}
&\Pi_2(\{f_A, f_B\}, \{f_A, f_B\})\\
&\quad = \pi_2(t_1, (2, f_B))\mu(t_1) + \pi_2(t_2, (1, f_A))\mu(t_2) + \pi_2(t_3, (2, f_A))\mu(t_3).
\end{aligned}$$

Recalling that $\mu(t_1) = \mu(t_2) = \mu(t_3) = 1/3$ and using the numbers in expressions (30) and (31), we obtain

$$\Pi_1(\{f_A, f_B\}, \{f_A, f_B\}) = 2,$$

and

$$\Pi_2(\{f_A, f_B\}, \{f_A, f_B\}) = 3.$$

7.1.3. The payoff matrix

Computing the expected profits for all possible catalog profiles, we obtain the following payoff matrix.

Firm 1 ↓ / Firm 2 →	$\{0\}$	$\{f_A\}$	$\{f_B\}$	$\{f_A, f_B\}$
$\{0\}$	(0,0)	(0,2)	(0,6)	(0,5)
$\{f_A\}$	(4,0)	(4,1)	(2,4)	(2,3)
$\{f_B\}$	(2,0)	(1,2)	(1,4)	(0,5)
$\{f_A, f_B\}$	(5,0)	(4,1)	(3,2)	(2,3)*

Payoff Matrix for Example 1 (32)

Note that $\Pi_1(\{f_A, f_B\}, \{f_A, f_B\})$ and $\Pi_2(\{f_A, f_B\}, \{f_A, f_B\})$ computed above give us the entries, (2, 3), for the fourth row and fourth column of the payoff matrix—the expected profits which result if firm 1 offers catalog $C_1 = \{f_A, f_B\}$ and firm 2 offers catalog $C_2 = \{f_A, f_B\}$.

In this catalog game,

$$(C_1, C_2) = (\{f_A, f_B\}, \{f_A, f_B\}),$$

with corresponding payoff (2, 3), is the unique Nash equilibrium catalog profile (marked by * in the payoff matrix above). Corresponding to this catalog profile there is only one competitively viable direct mechanism. It is given by

$$(i(t), f(t)) = \begin{cases} (2, f_B) \text{ if } t = t_1, \\ (1, f_A) \text{ if } t = t_2, \\ (2, f_A) \text{ if } t = t_3. \end{cases} \tag{33}$$

For $j = 1, 2$, let

$$T_j = \{t \in T : i(t) = j\}.$$

The *unique, minimal* catalog profile implementing the mechanism in expression (33) is given by

$$(\{f(t) : t \in T_1\}, \{f(t) : t \in T_2\}).$$

By inspection of expression (33), we obtain

$$\{f_A\} = \{f(t) : t \in T_1\},$$

and

$$\{f_A, f_B\} = \{f(t) : t \in T_2\}.$$

Thus, the unique, minimal implementing catalog profile corresponding to the mechanism in expression (33) is

$$(\{f_A\}, \{f_A, f_B\}),$$

and this is not a Nash equilibrium for the catalog game with payoff matrix given in expression (32). We can conclude therefore that the Nash catalog profile,

$$(\{f_A, f_B\}, \{f_A, f_B\}),$$

has no corresponding Nash mechanism, and cannot be implemented via any competitively viable direct mechanism. Note that contract f_B in firm 1's catalog is not chosen by any agent type.

7.2. Example 2

7.2.1. Altering Example 1

Suppose we change the definition of firm 2's profit function as follows:

$$\left.\begin{array}{l} \text{For all } t \in T, \\ \pi_2(t,(2,f_B)) = 3 \quad \text{and} \quad \pi_2(t,(2,f_A)) = 3, \\ \text{while} \\ \pi_2(t,(1,f_A)) = \pi_2(t,(1,f_B)) = 0, \\ \text{and} \\ \pi_2(t,(0,0)) = \pi_2(t,(1,0)) = \pi_2(t,(2,0)) = 0. \end{array}\right\} \tag{34}$$

Thus, in this example, contracts f_A and f_B are equally profitable for firm 2. Leaving all the other ingredients in Example 1 unchanged, with this change in firm 2's profit function, we obtain the following payoff matrix.

Firm 1 ↓ / Firm 2 →	$\{0\}$	$\{f_A\}$	$\{f_B\}$	$\{f_A, f_B\}$
$\{0\}$	(0,0)	(0,2)	(0,3)	(0,3)
$\{f_A\}$	(4,0)	(4,1)	(2,2)	(2,2)*
$\{f_B\}$	(2,0)	(1,2)	(1,2)	(0,3)
$\{f_A, f_B\}$	(5,0)	(4,1)	(3,1)	(2,2)*

Payoff Matrix for Example 2 (35)

7.2.2. Nash equilibria

In this catalog game, there are two Nash equilibrium catalog profiles (each marked by * in the payoff matrix above),

$$(\{f_A, f_B\}, \{f_A, f_B\}) \text{ with corresponding payoff } (2, 2),$$

and

$$(\{f_A\}, \{f_A, f_B\}) \text{ with corresponding payoff } (2, 2).$$

As in Example Examples, the only competitively viable direct mechanism corresponding to Nash catalog profile $(\{f_A, f_B\}, \{f_A, f_B\})$ is given by

$$(i(t),\, f(t)) = \begin{cases} (2,\, f_{\mathrm{B}}) & \text{if } t = t_1, \\ (1,\, f_{\mathrm{A}}) & \text{if } t = t_2, \\ (2,\, f_{\mathrm{A}}) & \text{if } t = t_3, \end{cases} \tag{36}$$

and this mechanism, as we showed in Example Examples, cannot be used to implement this catalog profile. However, observe that this mechanism *is* a Nash mechanism for the other Nash catalog profile, $(\{f_{\mathrm{A}}\}, \{f_{\mathrm{A}}, f_{\mathrm{B}}\})$. To see this recall from Example 1 that

$$\{f_{\mathrm{A}}\} = \{f(t) : t \in T_1\}$$

and

$$\{f_{\mathrm{A}}, f_{\mathrm{B}}\} = \{f(t) : t \in T_2\},$$

where

$$T_j = \{t \in T : i(t) = j\}.$$

Thus, the *unique*, *minimal* catalog profile implementing the competitively viable direct mechanism in expression (36) is given by

$$(\{f(t) : t \in T_1\}, \{f(t) : t \in T_2\}) = (\{f_{\mathrm{A}}\}, \{f_{\mathrm{A}}, f_{\mathrm{B}}\}),$$

and thus in Example 2, this mechanism is a Nash mechanism.

8. Nonlinear pricing games and Nash equilibrium

In this section, we consider the problem of existence of Nash equilibrium for competitive nonlinear pricing games. Due to the unusual nature of each firm's strategy set, the existence problem is difficult. In particular, each firm must choose not only a pricing function (i.e. a function which assigns to each product a price) but also a product line (i.e. a set of products upon which the pricing function is defined). By the competitive taxation principle any non-cooperative game played over implementable nonlinear pricing schedules is strategically equivalent to a much simpler game played over product–price catalogs. Thus, in order to resolve the existence issue for nonlinear pricing games, we need only consider catalog games played over product–price catalogs. However, as mentioned previously, even for the simpler catalog game, proving existence of a Nash equilibrium is not straightforward. First, because the space of catalogs (a compact metric space) is not a vector space, the usual method of proving existence via a fixed point argument is not available for catalog games. Thus, in order to address the existence question, we must introduce mixed strategies (or probabilistic strategies) over catalogs and consider the mixed extension of the catalog game.[31] Second,

[31] It would be very difficult to define the mixed extension of the game played over nonlinear pricing schedules. This would involve defining, for each firm j, probability measures on the set of pairs $(X_j, p_j(\cdot))$. The competitive taxation principle, by making it possible to move from nonlinear pricing schedules to catalogs and back again, allows us to consider, without loss of generality, the simpler game played over catalogs, where it is much easier to define the mixed extension.

because in general the agent's best response mapping is upper semicontinuous in catalog profiles (rather than continuous), and because each firm's profit function is upper semicontinuous in contracts (rather than continuous), except in the case of finitely many contracts, the mixed extension of the catalog game is oftentimes discontinuous. Our proof of the existence of a Nash equilibrium for the mixed extension of the catalog game rests upon a recent result due to Reny (1999) on existence of Nash equilibria in discontinuous games. In particular, for mixed catalog games satisfying Reny's condition of payoff security, we deduce the existence of a Nash equilibrium via Reny's existence result for payoff secure, upper semicontinuous games. Then, applying our existence result for mixed catalog games to mixed catalog games played over product–price catalogs, we are able deduce, via the competitive taxation principle, the existence of a Nash equilibrium for the mixed extension of the nonlinear pricing game. We also address the issue of which catalog games possess payoff secure mixed extensions. We begin by introducing the notion of uniform payoff security, a condition implying payoff security. We then identify a large class of competitive nonlinear pricing games for which the corresponding catalog games satisfy uniform payoff security. We conclude by showing that the mixed extension of any uniformly payoff secure catalog game is payoff secure.

8.1. Mixed catalog strategies

For each firm $i = 1, 2, \dots, m$, let $\Delta(P_f(K_j))$ denote the set of all probability measures defined on the feasible set of catalogs, $P_f(K_j)$. The strategy set $\Delta(P_f(K_j))$ is the jth firm's mixed (or probabilistic) catalog strategy set. A mixed catalog strategy for firm j is a probability measure $\lambda_j \in \Delta(P_f(K_j))$ having the following interpretation: if firm j chooses strategy $\lambda_j \in \Delta(P_f(K_j))$, then given any Borel measurable subset of catalogs, $\boldsymbol{E} \in B(P_f(K_j))$, the probability that a catalog contained in $\boldsymbol{E}$ will be selected under strategy λ_j is $\lambda_j(\boldsymbol{E})$. Since the feasible set of catalogs, $P_f(K_j)$, equipped the Hausdorff metric, is a compact metric space, the mixed catalog strategy set $\Delta(P_f(K_j))$ is convex, compact, and metrizable for the topology of weak convergence of measures (see Aliprantis and Border (1999), Chapter 14).

If firms choose mixed strategy profile

$$(\lambda_1, \dots, \lambda_m) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m)),$$

then the jth firm's expected payoff is given by

$$F_j(\lambda_1, \dots, \lambda_m) := \int_{P_f(K_1)\times\cdots\times P_f(K_m)} \Pi_j(C_1, \dots, C_m)\lambda_1(\mathrm{d}C_1) \cdot \cdots \cdot \lambda_m(\mathrm{d}C_m). \tag{37}$$

Following the terminology of Reny (1999), the game $(\Delta(P_f(K_j)), F_j)_{j=1}^m$ represents the mixed extension of the underlying catalog game $(P_f(K_j), \Pi_j)_{j=1}^m$. We shall refer to the game $(\Delta(P_f(K_j)), F_j)_{j=1}^m$ as the mixed catalog game. Note that for each firm j the expected payoff function,

$$(\lambda_1, \dots, \lambda_m) \to F_j(\lambda_1, \dots, \lambda_m),$$

is multilinear and upper semicontinuous on the strategy space $\Delta(P_f(K_1))\times\cdots\times\Delta(P_f(K_m))$ (see Aliprantis and Border (1999), p. 479, Theorem 14.5). Also, note that if the underlying contract set K is finite, then each firm's expected payoff function is multilinear and continuous.

8.2. Mixed catalog games, payoff security, and the existence of Nash Equilibrium

Definition (Nash equilibrium for mixed catalog games). A strategy profile

$$(\lambda_1^*, \dots, \lambda_m^*) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m))$$

is a Nash equilibrium for the mixed catalog game

$$(\Delta(P_f(K_j)), F_j)_{j=1}^m$$

if for all $j = 1, 2, \dots, m$

$$F_j(\lambda_j^*, \lambda_{-j}^*) \geq F_j(\lambda_j, \lambda_{-j}^*), \quad \text{for all } \lambda_j \in \Delta(P_f(K_j)). \tag{38}$$

If

$$(\lambda_1^*, \dots, \lambda_m^*) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m)),$$

is a Nash equilibrium for the mixed catalog game $(\Delta(P_f(K_j)), F_j)_{j=1}^m$, then for all $j = 1, \dots, m$,

$$\max_{C_j\in P_f(K_j)} F_j(C_j, \lambda_{-j}^*) = \max_{\lambda_j\in\Delta(P_f(K_j))} F_j(\lambda_j, \lambda_{-j}^*) = F_j(\lambda_j^*, \lambda_{-j}^*), \tag{39}$$

where

$$F_j(C_j, \lambda_{-j}^*) := \int_{C_{-j}} \Pi_j(C_j, C_{-j})\lambda_{-j}^*(\mathrm{d}C_{-j}).$$

Thus, for each $j = 1, \dots, m$,

$$\lambda_j^*(\arg\max_{C_j\in P_f(K_j)} F_j(C_j, \lambda_{-j}^*)) = 1. \tag{40}$$

From the perspective of the jth firm, if the mixed catalog strategy, λ_j^*, is the jth firm's part of a Nash equilibrium, then it will choose with probability 1 a catalog that maximizes $F_j(C_j, \lambda_{-j}^*)$ over the jth firms feasible set of catalogs, $P_f(K_j)$—and thus will choose an optimal catalog given the mixed strategies, λ_{-j}^*, of the other firms.[32]

[32] Here,

$$(\lambda_j, \lambda_{-j}^*) = (\lambda_1^*, \dots, \lambda_{j-1}^*, \lambda_j, \lambda_{j+1}^*, \dots, \lambda_m^*),$$

$$C_{-j} = (C_1, \dots, C_{j-1}, C_{j+1}, \dots, C_m),$$

and

$$\lambda_{-j}(\mathrm{d}C_{-j}) = \prod_{i\neq j} \lambda_i(\mathrm{d}C_i).$$

We begin by stating a condition, called payoff security, introduced by Reny (1999). This condition is crucial to the existence of a Nash equilibrium in games with discontinuous payoffs such as the mixed catalog game.

8.2.1. Payoff security (mixed strategies)

We say that the mixed catalog game $(\Delta(P_f(K_j)), F_j)_{j=1}^m$, *is payoff secure if for every mixed strategy profile*

$$(\lambda'_1, \ldots, \lambda'_m) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m)),$$

and every $\varepsilon > 0$ *there exists a* $\delta > 0$ *and a mixed strategies* $\lambda^*_j \in \Delta(P_f(K_j))$, $j = 1, \ldots, m$, *such that*

$$F_j(\lambda^*_j, \lambda_{-j}) \geq F_j(\lambda'_j, \lambda'_{-j}) - \varepsilon.$$

for all $(m-1)$*-tuples of mixed strategies* $\lambda_{-j} \in B_\delta(\lambda'_{-j})$.

Here, $B_\delta(\lambda'_{-j})$ is an open ball of radius δ centered at λ'_{-j} in the metric space of probability measures,

$$\Delta_{-j}(P_f(K_{-j})) := \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_{j-1})) \times \Delta(P_f(K_{j+1})) \times \cdots \times \Delta(P_f(K_m)).$$

Thus, the mixed catalog game is payoff secure if starting at any mixed strategy profile, $(\lambda'_1, \ldots, \lambda'_m)$, each firm j has a mixed strategy, λ^*_j, it can move to in order to preserve an expected payoff of $F_j(\lambda'_j, \lambda'_{-j}) - \varepsilon$ if other firms begin to deviate from their mixed strategies, λ'_{-j}, by moving to other mixed strategies in a neighborhood of λ'_{-j}. Note that if the underlying contract space K is finite, then the mixed catalog game, $(\Delta(P_f(K_j)), F_j)_{j=1}^m$, automatically satisfies payoff security. Moreover, the mixed catalog game will satisfy payoff security if for each firm j and each mixed strategy $\lambda_j \in \Delta(P_f(K_j))$, the function $F_j(\lambda_j, \cdot)$ is continuous on $\Delta_{-j}(P_f(K_{-j}))$.

The proof of our existence result follows directly from Proposition 5.1 and Corollary 5.2 in Reny (1999).

Theorem 4 (Existence of Nash equilibrium). *Suppose assumptions* (A-1)–(A-4) *hold. If the mixed catalog game* $(\Delta(P_f(K_j)), F_j)_{j=1}^m$, *is payoff secure, then it has a Nash equilibrium* $(\lambda^*_1, \ldots, \lambda^*_m) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m))$.

It follows from the competitive taxation principle that each nonlinear pricing schedule $(X_j, p_j(\cdot))$, is uniquely indexed by a product–price catalog $C_j \in P_f(K_j)$. Thus, the mixed catalog game, $(\Delta(P_f(K_j)), F_j)_{j=1}^m$, played over product–price catalogs can be viewed as the mixed extension of the nonlinear pricing game played over the index set, $P_f(K_1) \times \cdots \times P_f(K_m)$. If under the Nash equilibrium profile of mixed strategies,

$$(\lambda^*_1, \ldots, \lambda^*_m),$$

product–price catalog profile $(C_1, \ldots, C_m)$ is chosen, then this is equivalent to choosing the profile of nonlinear pricing schedules,

$$((X_1, p_1(\cdot)), \ldots, (X_m, p_m(\cdot))),$$

uniquely indexed by $(C_1, \dots, C_m)$. Moreover, from the perspective of the jth firm any catalog, say C_j^*, selected under mixed strategy $\lambda_j^* \in \Delta(P_f(K_j))$ is such that

$$\begin{aligned} F_j(C_j^*, \lambda_{-j}^*) &= \max F_j(C_j, \lambda_{-j}^*) C_j \in P_f(K_j) \\ &= \max_{\lambda_j \in \Delta(P_f(K_j))} F_j(\lambda_j, \lambda_{-j}^*) = F_j(\lambda_j^*, \lambda_{-j}^*). \end{aligned}$$

Thus, if $(X_j^*, p_j^*(\cdot))$ is the nonlinear pricing schedule uniquely indexed by C_j^*, then $(X_j^*, p_j^*(\cdot))$ is optimal (in an expected sense) against all the nonlinear pricing schedules which might emerge under the mixed catalog strategies, λ_{-j}^*, adopted by the jth firm's competitors.

8.3. A class of payoff secure competitive nonlinear pricing games

In this subsection, we introduce the notion of uniform payoff security and we identify a class of product–price catalog games such that each game in this class is uniformly payoff secure. We then show that the mixed extension of any catalog game from this class is payoff secure. We begin by defining the notion of uniform payoff security.

8.3.1. Uniform payoff security (pure strategies)

We say that the catalog game, $(P_f(K_j), \Pi_j)_{j=1}^m$, is uniformly payoff secure if for every catalog C_j^* and $\varepsilon > 0$, there exists a $\delta > 0$ and a catalog $\hat{C}_j \in P_f(K_j)$, $j = 1, \dots, m$, such that

$$\Pi_j(\hat{C}_j, C'_{-j}) \geq \Pi_j(C_j^*, C_{-j}) - \varepsilon.$$

for all $(m-1)$-tuples of catalogs C'_{-j} and C_{-j} such that $h_{\boldsymbol{P}_{-j}}(C'_{-j}, C_{-j}) < \delta$.

Here, $h_{\boldsymbol{P}_{-j}}(C'_{-j}, C_{-j})$ is given by

$$h_{\boldsymbol{P}_{-j}}(C'_{-j}, C_{-j}) := \max\{h(C'_i, C_i) : i \neq j\},$$

and defines a metric on the space of partial catalog profiles,

$$P_f(K_{-j}) := P_f(K_1) \times \cdots \times P_f(K_{j-1}) \times P_f(K_{j+1}) \times \cdots \times P_f(K_m).$$

It is easy to see that if a catalog games satisfies uniform payoff security, then it automatically satisfies payoff security.

Next, assume that the agent's utility is given by

$$v(t, (i, x, p)) := u(t, i, x) - p, \tag{41}$$

and that the jth firm's profit function is given by

$$\pi_j(t, i, x, p) := (p - c_j(x)) I_j(i), \tag{42}$$

where

$$I_j(i) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

We will assume that $u(\cdot, \cdot, \cdot)$ is continuous in (i, x) and measurable in t, and that $u(t, i, 0) < u(t, 0, 0)$. We will also assume that for each $i \in I$, the family of functions

$$U_i = \{u(t, i, \cdot) : t \in T\},$$

is equicontinuous. Thus, given any $\varepsilon > 0$, there is a $\dot{\delta} > 0$ then $|u(t, i, x) - u(t, i, x')| < \varepsilon$ for all $t \in T$. Moreover, we will assume that each firm's cost function, $c_j(\cdot)$ is bounded and lower semicontinuous. Thus, the agent's utility functions satisfies (A-3)* and each firm's profit function satisfies (A-4). Note that each firm's profit function does not depend directly on the agent's type.

We now have our main result on uniform payoff security for the class of product–price catalog games described above.

Theorem 5 (Price-linear, product–price catalog games are uniformly payoff secure). *If* $(P_f(K_j), \Pi_j)_{j=1}^m$ *is a product–price catalog game satisfying assumptions* (A-1), (A-2)*, (A-3)*, *and* (A-4) *such that the agent's utility function is given by* (41) *and each firm's profit function is given by* (42), *then* $(P_f(K_j), \Pi_j)_{j=1}^m$ *is uniformly payoff secure.*

In fact, it turns out that the mixed extension $(\Delta(P_f(K_j)), F_j)_{j=1}^m$ of any uniformly payoff secure catalog game $(P_f(K_j), \Pi_j)_{j=1}^m$ is payoff secure. Thus, the mixed extension of any price-linear, product–price catalog game satisfying (A-1), (A-2)*, (A-3)*, and (A-4) is payoff secure, and therefore, by Theorem 4, has a Nash equilibrium.

Theorem 6 (Price-linear, mixed product–price catalog games are payoff secure). *If* $(P_f(K_j), \Pi_j)_{j=1}^m$ *is a product–price catalog game satisfying assumptions* (A-1), (A-2)*, (A-3)*, *and* (A-4) *such that the agent's utility function is given by* (41) *and each firm's profit function is given by* (42), *then the mixed extension* $(\Delta(P_f(K_j)), F_j)_{j=1}^m$, *of* $(P_f(K_j), \Pi_j)_{j=1}^m$ *is payoff secure.*

Under our definition of expected potential profit (see expressions (16) and (17) above), each firm in computing expected potential profit *assumes* that whenever the agent is indifferent between the firm's catalog and the catalogs of one or more other firms, the agent will choose the firm's catalog (i.e. each firm assumes that ties will be broken in the firm's favor). Other tie breaking rules are possible. For example, we might assume that in the case of ties the agent chooses between firms randomly, with equal probability. However, under such a tie breaking rule, upper semicontinuity of the expected potential profit function is lost. With the loss of upper semicontinuity, payoff security is no longer sufficient to guarantee the existence of a Nash equilibrium for the mixed catalog game via Reny's result (see Reny (1999), Corollary 5.2). Instead, we must show that the mixed catalog game is *better-reply secure*—a notion also introduced in Reny (1999). The mixed catalog game, $(\Delta(P_f(K_j)), F_j)_{j=1}^m$, is said to be better-reply secure if whenever

$$((\lambda'_1, \ldots, \lambda'_m), (F_1(\lambda'_1, \ldots, \lambda'_m), \ldots, F_1(\lambda'_1, \ldots, \lambda'_m)))$$

is in the closure of the graph of

$$(\lambda_1, \ldots, \lambda_m) \to F_j(\lambda_1, \ldots, \lambda_m),$$

and

$$(\lambda'_1, \dots, \lambda'_m) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m))$$

is *not* a Nash equilibrium, then some firm j can secure a payoff strictly greater than

$$F_j(\lambda'_1, \dots, \lambda'_m).$$

Thus, for some $\varepsilon > 0$, there exists $\lambda''_j \in \Delta(P_f(K_j))$ and $\delta > 0$ such that

$$F_j(\lambda''_j, \lambda_{-j}) \geq F_j(\lambda'_1, \dots, \lambda'_m) + \varepsilon$$

for all $(m-1)$-tuples of mixed strategies $\lambda_{-j} \in B_\delta(\lambda'_{-j})$. As before, $B_\delta(\lambda'_{-j})$ is an open ball of radius δ centered at λ'_{-j} in the metric space of probability measures,

$$\Delta_{-j}(P_f(K_{-j})) := \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_{j-1})) \times \Delta(P_f(K_{j+1})) \times \cdots \\ \times \Delta(P_f(K_m)).$$

In Monteiro and Page (2002), we show that under the random tie-breaking rule described above, any mixed catalog game $(\Delta(P_f(K_j)), F_j)_{j=1}^m$, corresponding to a price-linear, product–price catalog game $(P_f(K_j), \Pi_j)_{j=1}^m$, is better-reply secure.

Acknowledgements

The current version of the paper was presented at Advances in Game Theory and Related Topics (Hilvarenbeek, 23–25 June 2002), PET02 (Paris, 4–6 July 2002), The Sixth International Meeting of the Society for Social Choice and Welfare (Caltech, 11–14 July 2002), and The First Brasilian Workshop of the Game Theory Society (Sao Paulo, 28 July–3 August 2002). The authors thank the participants in these conferences for many helpful comments. The earlier version of this paper was completed while Page was visiting CERMSEM at the University of Paris 1 and the Warwick Center for Public Economics at the University of Warwick. Page gratefully acknowledges the support and hospitality of these institutions. Monteiro gratefully acknowledges the support of CNPq. Both authors thank Tom Gresik and seminar participants at Paris 1, Warwick, Washington University, and Tulane University for many helpful comments. Both authors are especially grateful to an anonymous referee whose thoughtful comments led to significant improvements in the paper.

Appendix

A.1. Some definitions

A.1.1. Measurability

A function $f(\cdot)$ defined on the measurable space $(Z, \mathfrak{I})$ taking values in some closed bounded interval $[L, H]$ of the real numbers is $\mathfrak{I}$-measurable on Z if and only if for each Borel measurable subset E of $[L, H]$ (i.e. for each $E \in B([L, H])$)

$$\{z \in Z : f(z) \in E\} \in \mathfrak{I}.$$

Given $(i, f) \in \mathbb{K}$, the real-valued function $v(\cdot, i, f)$ is $B(T)$-measurable on T if and only if for each Borel measurable subset, E, of the real numbers (i.e. $E \in B(R)$),

$$\{t \in T : v(t, i, f) \in E\} \in B(T).$$

The real-valued profit function $\pi_j(\cdot, \cdot, \cdot)$ defined on $T \times \mathbb{K}$ is $B(T) \times B(\mathbb{K})$-measurable if and only if for each Borel measurable subset, E, of the real numbers,

$$\{(t, i, f) \in T \times \mathbb{K} : \pi_j(t, i, f) \in E\} \in B(T) \times B(\mathbb{K}).$$

Here, $B(T) \times B(\mathbb{K})$ denotes the product σ-field in $T \times \mathbb{K}$.

A direct contracting mechanism

$$t \to (i(t), f(t))$$

is a $(B(T), B(\mathbb{K}))$-measurable if and only if for each subset $E \in B(\mathbb{K})$,

$$\{t \in T : (i(t), f(t)) \in E\} \in B(T).$$

A.1.2. The Hausdorff metric

Let d be the metric on K and define $d(f, C) := \inf_{f' \in C} d(f, f')$ for $f \in K$ and C a nonempty closed subset of K. For $i = 0, 1, \ldots, m$, and $P_f(K_i)$ the collection of all nonempty, closed subsets of $K_i \subseteq K$ the Hausdorff metric h on $P_f(K_i)$ is defined as follows:

$$h(C, C') := \max\{\sup_{f \in C} d(f, C'), \sup_{f' \in C'} d(f', C)\}, \quad \text{for } C \quad \text{and } C' \quad \text{in } P_f(K_i).$$

For $i = 0, 1, \ldots, m$ convergence in catalogs, i.e. convergence in the compact metric space $(P_f(K_i), h)$, can be characterized via the notions of limes inferior and limes superior. Let $\{C_n\}_n$ be a sequence in $P_f(K_i)$. The limes inferior of this sequence, denoted by $\mathrm{Li}(C_n)$, is defined as follows: $f \in \mathrm{Li}(C_n)$ if and only if there is a sequence $\{f_n\}_n$ in K_i such that $f_n \in C_n$ for all n and $f_n \to f$. The limes superior, denoted by $\mathrm{Ls}(C_n)$, is defined as follows: $f \in \mathrm{Ls}(C_n)$ if and only if there is a subsequence $\{f_{n_k}\}_k$ in K_i such that $f_{n_k} \in C_{n_k}$ for all k and $f_{n_k} \to f$. A catalog $C \in P_f(K_i)$ is said to be the limit of catalogs $\{C_n\}_n$ if $\mathrm{Ls}(C_n) = C = \mathrm{Li}(C_n)$. Moreover, $h(C_n, C) \to 0$ (i.e. the sequence $\{C_n\}_n$ converges to $C \in P_f(K_i)$ under the Hausdorff metric h) if and only if $\mathrm{Ls}(C_n) = C = \mathrm{Li}(C_n)$ (see Aliprantis and Border (1999), Sections 3.14–3.16).

A.1.3. Measurability and upper semicontinuity of Φ

The mapping $\Phi(\cdot, \cdot, \ldots, \cdot)$ defined on $T \times \boldsymbol{P}$ is $B(T) \times B(\boldsymbol{P})$-measurable if and only if for each closed subset F of $\mathbb{K}$ the set,

$$\{(t, C_1, \ldots, C_m) \in T \times \boldsymbol{P} : \Phi(t, C_1, \ldots, C_m) \cap F \neq \emptyset\},$$

is contained in the product σ-field, $B(T) \times B(\boldsymbol{P})$. For each $t \in T$, the best response mapping, $\Phi(t, \cdot, \ldots, \cdot)$, is $h_{\boldsymbol{P}}$-upper semicontinuous on $\boldsymbol{P}$ if and only if for each closed subset F of $\mathbb{K}$ the set,

$$\{(C_1, \ldots, C_m) \in \boldsymbol{P} : \Phi(t, C_1, \ldots, C_m) \cap F \neq \emptyset\},$$

is $h_{\boldsymbol{P}}$-closed in $\boldsymbol{P}$.

A.1.4. Continuity and semicontinuity in metric spaces

Let X and Y be metric spaces and let $g(\cdot) : X \to Y$ be a function. The function $g(\cdot)$ is continuous if and only if for each sequence $\{x^n\}_n$ in X converging to $x \in X$,

$$\lim_n g(x^n) = g(x).$$

If $g(\cdot) : X \to [-\infty, \infty]$ is a mapping from X into the extended reals, then $g(\cdot)$ is upper semicontinuous if and only if for each sequence $\{x^n\}_n$ in X converging to $x \in X$,

$$\limsup_n g(x^n) \leq g(x),$$

or equivalently, if and only if for each $\alpha \in (-\infty, \infty)$ the set,

$$\{x \in X : g(x) \geq \alpha\},$$

is closed.

The function $g(\cdot) : X \to [-\infty, \infty]$ is lower semicontinuous if and only if for each sequence $\{x^n\}_n$ in X converging to $x \in X$,

$$\liminf_n g(x^n) \geq g(x),$$

or equivalently, if and only if for each $\alpha \in (-\infty, \infty)$ the set,

$$\{x \in X : g(x) \leq \alpha\},$$

is closed.

A.2. Proof of Theorem 1 (The delegation principle)

$1 \Rightarrow 2$ Let $(i(\cdot), f(\cdot))$ be a competitively viable direct mechanism. For $i = 0, 1, \ldots, m$, define

$$T_i := \{t \in T : i(t) = i\}, \tag{43}$$

and for i such that $T_i \neq \emptyset$, let

$$R_i := \{f \in K : f = f(t), \quad \text{for some } t \in T_i\} \tag{44}$$

Now define

$$C_i := \begin{cases} \mathrm{cl} R_i, & \text{if } T_i \neq \emptyset, \\ \{0\}, & \text{if } T_i = \emptyset, \end{cases} \tag{45}$$

where cl denotes closure.

It follows directly from definition (45) that

$$(i(t), f(t)) \in \Gamma(C_0, C_1, \ldots, C_m), \quad \text{for all } t \in T.$$

Thus, we have

$$v(t, i(t), f(t)) \leq v^*(t, C_1, \ldots, C_m), \quad \text{for all } t \in T.$$

Suppose now that for some agent type t'

$$v(t', i(t'), f(t')) < v^*(t', C_1, \ldots, C_m). \tag{46}$$

Inequality (46) implies that for agent type t', there is a 2-tuple

$$(i', f') \in \Gamma(C_0, C_1, \ldots, C_m)$$

such that

$$v(t', i(t'), f(t')) < v(t', i', f'). \tag{47}$$

Thus, $f' \in C_{i'}$ where $C_{i'}$ is as defined in (45). By assumption (A-3)(iii), if $f' = 0$, then we have a contradiction since

$$v(t', i(t'), f(t')) \geq v(t', 0, 0) > v(t', i', 0).$$

Suppose now that $f' \neq 0$. By the definition of the catalog (45) and the continuity of utility $v(t', \cdot, \cdot)$ on $\mathbb{K}$ (see (A-3)(i)), there exits some agent type $\bar{t} \in T_{i'}$ such that $(i(\bar{t}), f(\bar{t}))$ is sufficiently close to (i', f') so that

$$v(t', i(t'), f(t')) < v(t', i(\bar{t}), f(\bar{t})). \tag{48}$$

But inequality (48) contradicts the assumed competitive viability of the mechanism $(i(\cdot), f(\cdot))$. We must conclude, therefore, that

$$v(t, i(t), f(t)) = v^*(t, C_1, \ldots, C_m), \quad \text{for all } t \in T,$$

implying that

$$(i(t), f(t)) \in \Phi(t, C_1, \ldots, C_m), \quad \text{for all } t \in T,$$

where the C_i are the catalogs defined in (45).

Suppose now that for some other catalog profile $(C'_1, \ldots, C'_m) \in \boldsymbol{P}$, it true that

$$(i(t), f(t)) \in \Phi(t, C'_1, \ldots, C'_m), \quad \text{for all } t \in T.$$

We want to show that this implies that $C_j \subseteq C'_j$ for all $j = 1, \ldots, m$ such that $i(t) = j$ for some $t \in T$. Suppose not. Then for some $j' = 1, \ldots, m$, there exists $f' \in C_{j'}$ such that $f' \notin C'_{j'}$. Since $C'_{j'}$ is closed, given the construction of the catalog $C_{j'}$ (see (45)), there exists $t' \in T_{j'}$ such that $f(t') \in C_{i(t')} = C_{j'}$, sufficiently close to $f' \in C_{j'}$ is also not contained in $C'_{j'}$ (i.e. $f(t') \notin C'_{j'}$). But this implies that

$$(i(t'), f(t')) \notin \Phi(t', C'_1, \ldots, C'_m),$$

a contradiction.

The proof that $2 \Rightarrow 1$ is straightforward and thus omitted.

A.3. Proof of Theorem 2 (The implementation principle)

A.3.1. The existence of message selection

In this subsection, we identify a set of conditions (not the only set) sufficient to guarantee the existence of a message selection.

First, recall that a profile of indirect contracting mechanisms,

$$((M_1, g_1(\cdot)), \ldots, (M_m, g_m(\cdot))),$$

is said to be incentive compatible if and only if there exists a function,

$$t \to (i(t), m(t)),$$

called a message selection such that

(i) $t \to (i(t), g_{i(t)}(m(t)))$ is $(B(T), B(\mathbb{K}))$-measurable,
(ii) for all $t \in T$, $(i(t), m(t)) \in \Psi(M_0, M_1, \ldots, M_m)$, and
(iii) for all $t \in T$, $v(t, i(t), g_{i(t)}(m(t))) = \max\{v(t, i, g_i(m)) : (i, m) \in \Psi(M_0, M_1, \ldots, M_m)\}$.

To begin, assume that the space of all possible messages M is a metric space. Next, assume that the profile of indirect contracting mechanisms,

$$((M_1, g_1(\cdot)), \ldots, (M_m, g_m(\cdot))),$$

is such that (a) each firm's message space, M_j, is a compact subset of M, and (b) each firm's message function,

$$g_j(\cdot) : M_j \to K_j,$$

is upper semicontinuous. We shall refer to any profile of indirect contracting mechanisms satisfying (a) and (b) as an upper semicontinuous profile.

It follows from Theorem 2 in Himmelberg et al. (1976) that corresponding to any upper semicontinuous profile of indirect mechanisms there exists at least one $(B(T), B_\Psi)$-measurable message selection,

$$t \to (i(t), m(t)) \in \Psi(M_0, M_1, \ldots, M_m),$$

such that

$$v(t, i(t), g_{i(t)}(m(t))) = \max\{v(t, i, g_i(m)) : (i, m) \in \Psi(M_0, M_1, \ldots, M_m)\}.$$

Here, B_Ψ denotes the Borel σ-field in the compact metric space $\Psi(M_0, M_1, \ldots, M_m) \subseteq I \times M$. A message selection $(i(\cdot), m(\cdot))$ is $(B(T), B_\Psi)$-measurable if and only if for each subset $E \in B_\Psi$,

$$\{t \in T : (i(t), m(t)) \in E\} \in B(T).$$

Given any upper semicontinuous profile of indirect mechanisms the function,

$$t \to (i(t), g_{i(t)}(m(t))),$$

is $(B(T), B(\mathbb{K}))$-measurable for any $(B(T), B_\Psi)$-measurable message selection. Also, under assumptions (A-3), the function,

$$v(t, \cdot, g_{(\cdot)}(\cdot)) : \Psi(M_0, M_1, \dots, M_m) \to R,$$

is upper semicontinuous, and the function,

$$v(\cdot, \cdot, g_{(\cdot)}(\cdot)) : T \times \Psi(M_0, M_1, \dots, M_m) \to R,$$

is $B(T) \times B_\Psi$-measurable.

A.3.2. Proof of the implementation principle

Proof of part 1: let

$$((M_1, g_1(\cdot)), \dots, (M_m, g_m(\cdot))),$$

be a profile of incentive compatible indirect contracting mechanisms. For each $j = 1, \dots, m$, define

$$C_j := \mathrm{cl}\{g_j(m) : m \in M_j\}, \tag{49}$$

where "cl" denotes closure. It suffices to show that for all $t \in T$,

$$\begin{aligned} &\max\{v(t, i, f) : (i, f) \in \Gamma(C_0, C_1, \dots, C_m)\} \\ &\quad = \max\{v(t, i, g_i(m)) : (i, m) \in \Psi(M_0, M_1, \dots, M_m)\}. \end{aligned} \tag{50}$$

First observe that by the definition of the catalogs given in expression (49), we have for all $t \in T$,

$$\begin{aligned} &\max\{v(t, i, f) : (i, f) \in \Gamma(C_0, C_1, \dots, C_m)\} \\ &\quad \geq \max\{v(t, i, g_i(m)) : (i, m) \in \Psi(M_0, M_1, \dots, M_m)\}. \end{aligned}$$

Let $(i(\cdot), m(\cdot))$ be a *message selection* for $((M_1, g_1(\cdot)), \dots, (M_m, g_m(\cdot)))$. We have for all $t \in T$,

$$v(t, i(t), g_{i(t)}(m(t))) = \max\{v(t, i, g_i(m)) : (i, m) \in \Psi(M_0, M_1, \dots, M_m)\}.$$

Suppose that for some $t' \in T$ and $(i', f') \in \Gamma(C_0, C_1, \dots, C_m)$

$$v(t', i', f') > v(t', i(t'), g_{i(t')}(m(t'))).$$

Given the definition of catalogs in expression (49) and given the continuity of the utility function $v(t', \cdot, \cdot)$ on $\mathbb{K}$, it follows that for some $m' \in M_{i'}$,

$$v(t', i', g_{i'}(m')) > v(t', i(t'), g_{i(t')}(m(t'))),$$

a contradiction. Thus, equality (50) holds for all $t \in T$, and thus,

$$(i(\cdot), g_{i(\cdot)}(m(\cdot))) \in \sum(C_1, \dots, C_m).$$

Uniqueness of catalogs follows directly from the definition given in expression (49). Minimality follows from the proof of minimality given in the proof of the delegation principle.

Proof of part 2: let

$$(C_1, \ldots, C_m) \in \boldsymbol{P}$$

be a catalog profile. For each $j = 1, \ldots, m$, let the jth firm's indirect contracting mechanism be given by

$$(C_j, e_j(\cdot)),$$

where

$$e_j(\cdot) : C_j \to K_j$$

is simply the *identity* mapping. To complete the proof, observe that the set of all message selections from the indirect mechanism

$$((C_1, e_1(\cdot)), \ldots, (C_m, e_m(\cdot)))$$

is equal to

$$\sum(C_1, \ldots, C_m),$$

the set of all competitively viable direct mechanism corresponding to the catalog profile $(C_1, \ldots, C_m)$.

A.4. Proof of Theorem 3 (The competitive taxation principle)

Proof of part 1: for $j = 0, 1, 2, \ldots m$, let $\text{proj}_X C_j$ denote the projection of catalog C_j onto X. The closed set $X_j := \text{proj}_X C_j$ is the jth firm's product line. Now define the jth firm's nonlinear pricing schedule,

$$p_j(\cdot) : \text{proj}_X C_j \to D,$$

as follows: for $x \in \text{proj}_X C_j$ (i.e. x an element of the projection of catalog C_j onto X), define

$$p_j(x) := \min\{p \in D : (x, p) \in C_j\}. \tag{51}$$

For each firm $j = 1, 2, \ldots, m$, $p_j(x)$ is the minimum price the agent can pay for goods x from firm j given catalog C_j.[33] Since each consumer's utility is strictly decreasing in price (see (A-3)*(iv)), if the agent chooses to purchase goods x from firm $j's$ catalog (i.e. if the agent chooses to purchase $x \in \text{proj}_X C_j$), then the agent will also choose to pay price $p_j(x)$ (i.e. the agent will choose $(x, p_j(x)) \in C_j$ from firm j's catalog).

By construction, the nonlinear pricing schedule $(X_j, p_j(\cdot))$, corresponding to catalog C_j, is minimal in the sense that,

$$X_j := \text{proj}_X C_j \text{ and } p_j(x) := \min\{p \in D : (x, p) \in C_j\} \text{ for } x \in \text{proj}_X C_j.$$

[33] Recall that for $j = 0$, $p_0(0) = 0$.

Thus, for each j, $(X_j, p_j(\cdot))$ is unique for the catalog C_j. Moreover, by construction $p_j(\cdot)$ is lower semicontinuous and

$$\text{graph}\{p_j(\cdot)\} \subseteq K_j.$$

To show that for each competitively viable direct mechanism, $(i(\cdot), x(\cdot), p(\cdot)) \in \sum (C_1, \ldots, C_m)$, $x(t) \in X_{i(t)}$ and $p_{i(t)}(x(t)) = p(t)$ for all agent types $t \in T$, observe that for

$$(i(\cdot), x(\cdot), p(\cdot)) \in \sum(C_1, \ldots, C_m), (x(t), p(t)) \in C_{i(t)}, \quad \text{for all } t \in T.$$

Thus, $x(t) \in X_{i(t)}$ for all t, and by construction, the nonlinear pricing schedules are such that

$$p_{i(t)}(x(t)) \leq p(t), \quad \text{for all agent types } t \in T.$$

If for some agent type $t' \in T$,

$$p_{i(t')}(x(t')) < p(t'),$$

then, because the agent's utility is strictly decreasing in price, we would have for the type t' agent

$$v(t', i(t'), x(t'), p_{i(t')}(x(t'))) > v(t', i(t'), x(t'), p(t'))$$

i.e. the type t' agent would choose

$$(x(t'), p_{i(t')}(x(t'))) \in C_{i(t')},$$

rather than

$$(x(t'), p(t')) \in C_{i(t')}$$

as intended by the mechanism), and thus we would have

$$(i(t'), x(t'), p(t')) \notin \Phi(t', C_1, \ldots, C_m),$$

a contradiction.

Proof of part 2: let

$$((X_1, p_1(\cdot)), \ldots, (X_m, p_m(\cdot))),$$

be a profile of nonlinear pricing schedules. For each $j = 1, \ldots, m$, define

$$C_j := \text{cl}\{\text{graph}\{p_j(\cdot)\}\}, \tag{52}$$

where "cl" denotes closure. Now let $(i(\cdot), x(\cdot), p(\cdot)) \in \sum(C_1, \ldots, C_m)$. For all $t \in T$

$$(x(t), p(t)) \in C_{i(t)}.$$

Thus, for all $t \in T$, $x(t) \in X_{i(t)}$. To show that $p(t) = p_{i(t)}(x(t))$ for all $t \in T$, consider the following: First, observe that since the agent's utility is strictly decreasing in price, we have for all $t \in T$

$$p(t) = \min\{p \in D : (x(t), p) \in C_{i(t)}\},$$

otherwise, for some $t' \in T$, $(i(t'), x(t'), p(t')) \notin \Phi(t', C_1, \dots, C_m)$. Second, observe that by the definition of the catalog given in expression (52), $(x(t), p_{i(t)}(x(t))) \in C_{i(t)}$ for all $t \in T$. Thus, $p(t) \leq p_{i(t)}(x(t))$ for all $t \in T$. Now suppose that for some $t' \in T$, $p(t') < p_{i(t')}(x(t'))$. Since $(x(t'), p(t')) \in C_{i(t')}$, by the definition of the catalogs, there exists a sequence $\{(x^n, p^n)\}_n$ contained in $C_{i(t')} := \text{cl}\{\text{graph}\{p_{i(t')}(\cdot)\}\}$ such that $(x^n, p^n) \to (x(t'), p(t'))$. Again, by the definition of $C_{i(t')}$, $p_{i(t')}(x^n) = p^n$ for all n. Thus, we have

$$\lim_n p_{i(t')}(x^n) = \liminf_n p_{i(t')}(x^n) = p(t').$$

By the lower semicontinuity of $p_{i(t')}(\cdot)$ on $X_{i(t')}$, $p(t') \geq p_{i(t')}(x(t'))$. Thus, we have a contradiction. We must conclude, therefore, that $p(t) = p_{i(t)}(x(t))$ for all $t \in T$.

Uniqueness and minimality follow by construction.

A.5. Proof of Theorem 5 (Uniform payoff security for price-linear, product-price catalog games)

Let C_1^* be a catalog and $\epsilon > 0$ be given. By the equicontinuity of the sets U_j, there exists a $\delta > 0$ such that

$$\begin{aligned} &\max_{j\geq 2} d((x_j, p_j), (x_j^*, p_j^*)) < \delta \text{ implies that} \\ &|u(t, j, x_j) - p_j - (u(t, j, x_j^*) - p_j^*)| < \tfrac{1}{2}\varepsilon, \quad \text{for all } t \in T. \end{aligned}$$

Define a new catalog $C_{1\varepsilon}^*$ as follows:

$$C_{1\varepsilon}^* = \{(x, p - \varepsilon) : (x, p) \in C_1^*, p - c_1(x) \geq \epsilon\} \cup \{(0, 0)\}.$$

Let $(C_2, \dots, C_m)$ and be $(D_2, \dots, D_m)$ be partial catalog profiles such that

$$\max_{2\leq j\leq m} h(C_j, D_j) < \delta.$$

and consider firm 1. Let

$$(i(\cdot), x(\cdot), p(\cdot)) \in \sum(C_1^*, C_2, \dots, C_m)$$

be such that

$$\int_T (p(t) - c_1(x(t)) I_1(i(t)))\, d\mu(t) = \Pi_1(C_1^*, C_2, \dots, C_m).$$

Given the properties of the Hausdorff metric,

$$\begin{aligned} &h(C_j, D_j) < \delta, \quad \text{for } j \geq 2 \text{ implies that for any} \\ &(x_j, p_j)_{j\geq 2} \in (D_2, \dots, D_m), \text{ there exists a} \\ &(x_j^*, p_j^*)_{j\geq 2} \in (C_2, \dots, C_m) \text{ such that } \max_{j\geq 2}\{d((x_j, p_j), (x_j^*, p_j^*))\} < \delta. \end{aligned}$$

Moreover, since

$$(i(\cdot), x_1(\cdot), p(\cdot)) \in \sum(C_1^*, C_2, \dots, C_m),$$

for all $j \in I$ and for all $(x_j^*, p_j^*) \in D_j$, we have

$$u(t, j, x_j^*) - p_j^* \leq u(t, i(t), x(t)) - p(t) + \varepsilon = u(t, i(t), x(t)) - (p(t) - \varepsilon).$$

Therefore, for agent types

$$t \in T_1 := \{t \in T : i(t) = 1\}$$

we have

$$(i(t), x(t), p(t) - \varepsilon) \in \Phi(t, C_{1\varepsilon}^*, C_2, \ldots, C_m).$$

Thus,

$$\begin{aligned} \Pi_1(C_{1\varepsilon}^*, C_2, \ldots, C_m) &\geq \int_T (p(t) - \varepsilon - c_1(x(t))) I_1(i(t))\, \mathrm{d}\mu(t) \\ &\geq \int_T (p(t) - c_1(x(t))) I_1(i(t))\, \mathrm{d}\mu(t) - \varepsilon \\ &= \Pi_1(C_1^*, D_2, \ldots, D_m) - \varepsilon. \end{aligned}$$

A.6. Proof of Theorem 6 (Payoff security for price-linear, mixed product–price catalog games)

Let

$$(\lambda_1^*, \ldots, \lambda_m^*) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m)),$$

be any mixed strategy profile and consider firm 1. Firm 1's expected payoff is given by

$$F_1(\lambda_1^*, \ldots, \lambda_m^*) := \int_{P_f(K_1)\times\cdots\times P_f(K_m)} \Pi_1(C_1, \ldots, C_m)\lambda_1^*(\mathrm{d}C_1) \cdot \cdots \cdot \lambda_m^*(\mathrm{d}C_m).$$

Let $\varepsilon^* > 0$ be given, We will show that there exists a mixed strategy λ_1' and a $\delta > 0$ such that if

$$\mu_{-1} = \mu_2 \times \cdots \times \mu_m \in B_\delta(\lambda_{-1}^*)$$

then

$$F_1(\lambda_1', \mu_2, \ldots, \mu_m) \geq F_1(\lambda_1^*, \ldots, \lambda_m^*) - \varepsilon^*.$$

As before, $B_\delta(\lambda_{-1}^*)$ is an open ball of radius δ centered at λ_{-1}^* in the metric space of probability measures,

$$\Delta_{-1}(P_f(K_{-1})) := \Delta(P_f(K_2)) \times \cdots \times \Delta(P_f(K_m)).$$

First let $\varepsilon > 0$ be given and choose catalog $C_1^* \in P_f(K_1)$ so that

$$F_1(C_1^*, \lambda_2^*, \ldots, \lambda_m^*) \geq \max_{C_1 \in P_f(K_1)} F_1(C_1, \lambda_2^*, \ldots, \lambda_m^*) - \varepsilon.$$

Since

$$\max_{C_1 \in P_f(K_1)} F_1(C_1, \lambda_2^*, \ldots, \lambda_m^*) - \varepsilon \geq F_1(\lambda_1^*, \ldots, \lambda_m^*) - \varepsilon,$$

we have

$$F_1(C_1^*, \lambda_2^*, \ldots, \lambda_m^*) \geq F_1(\lambda_1^*, \ldots, \lambda_m^*) - \varepsilon. \tag{53}$$

By the uniform payoff security (UPS) property of the catalog game, given C_1^* and $\varepsilon > 0$, there exists $\delta > 0$ and $\hat{C}_1$ such for all C'_{-1} and C_{-1} with $C'_{-1} \in B_\delta(C_{-1})$

$$\Pi_1(\hat{C}_1, C'_{-1}) \geq \Pi_1(C_1^*, C_{-1}) - \varepsilon.$$

Now observe that the collection of open balls,

$$\{B_{\delta/2}(C_{-1})\}_{C_{-1} \in P_f(K_{-1})},$$

where each open ball, $B_{\delta/2}(C_{-1})$, has radius $\delta/2 > 0$ and is centered at C_{-1}, forms an open cover of the compact metric space $P_f(K_{-1})$. Thus, there is a finite subcover of $P_f(K_{-1})$, denoted by

$$\{B_{\delta/2}(C_{-1}^h)\}_{h \in H}$$

where $H = \{1, 2, \ldots, h^*\}$.

Because λ_{-1}^* is a finite measure on $P_f(K_{-1})$, for each h we can choose a radius r_h such that

$$\tfrac{1}{2}\delta < r_h < \delta,$$

such that

$$\lambda_{-1}^*(\partial B_{r_h}(C_{-1}^h)) = 0, \quad \text{for all } h.$$

Here, $\partial B_{r_h}(C_{-1}^h)$ denotes the boundary of the open ball $B_{r_h}(C_{-1}^h)$.

Now let $\bar{B}_{r_h}(C_{-1}^h)$ denote the closed ball of radius r_h centered at C_{-1}^h. Because $\bar{B}_{r_h}(C_{-1}^h) \subset B_\delta(C_{-1}^h)$, by UPS we have for each h

$$\Pi_1(\hat{C}_1, C'_{-1}) \geq \Pi_1(C_1^*, C_{-1}) - \varepsilon, \quad \text{for all } C'_{-1} \text{ and } C_{-1} \text{ contained in } \bar{B}_{r_h}(C_{-1}^h). \tag{54}$$

By the upper semicontinuity of Π_1 on the compact metric space,

$$P_f(K_1) \times \cdots \times P_f(K_m),$$

for each $h \in H$, there exists C_{-1}^{*h} such that

$$\Pi_1(C_1^*, C_{-1}^{*h}) = \max_{C_{-1} \in \bar{B}_{r_h}(C_{-1}^h)} \Pi_1(C_1^*, C_{-1}). \tag{55}$$

Therefore, by (54) and (55) we have

$$\Pi_1(\hat{C}_1, C'_{-1}) \geq \Pi_1(C_1^*, C_{-1}^{*h}) - \varepsilon, \quad \text{for all } C'_{-1} \in \bar{B}_{r_h}(C_{-1}^h). \tag{56}$$

Define

$$D^1 = \bar{B}_{r_1}(C^1_{-1}),\, D^2 = \bar{B}_{r_2}(C^2_{-1}) \setminus \bar{B}_{r_1}(C^1_{-1}), \ldots, D^{h^*} = \bar{B}_{r_{h^*}}(C^{h^*}_{-1}) \setminus \bigcup_{h=1}^{h^*} \bar{B}_{r_h}(C^h_{-1}).$$

Thus,

$$\begin{aligned}
&\int_{P_f(K_{-1})} \Pi_1(\hat{C}_1, C'_{-1})\,\mathrm{d}\mu_{-1}(C'_{-1}) \\
&\quad = \sum_{h=1}^{h^*} \int_{D^h} \Pi_1(\hat{C}_1, C'_{-1})\,\mathrm{d}\mu_{-1}(C'_{-1}) \\
&\quad \geq \sum_{h=1}^{h^*} \int_{D^h} (\Pi_1(C^*_1, C^{*h}_{-1}) - \varepsilon)\,\mathrm{d}\mu_{-1}(C_{-1}) \\
&\quad = \sum_{h=1}^{h^*} \Pi_1(C^*_1, C^{*h}_{-1})\mu_{-1}(D^h) - \varepsilon \\
&\quad = \sum_{h=1}^{h^*} \Pi_1(C^*_1, C^{*h}_{-1})\mu_{-1}(D^h) + (F_1(\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_m) - 2\varepsilon \\
&\qquad - (F_1(\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_m) - \varepsilon)) \\
&\quad \geq \sum_{h=1}^{h^*} \Pi_1(C^*_1, C^{*h}_{-1})\mu_{-1}(D^h) + (F_1(\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_m) - 2\varepsilon - F_1(C^*_1, \lambda^*_2, \ldots, \lambda^*_m)) \\
&\quad = \sum_{h=1}^{h^*} \Pi_1(C^*_1, C^{*h}_{-1})\mu_{-1}(D^h) \\
&\qquad + \left(F_1(\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_m) - 2\varepsilon - \sum_{h=1}^{h^*} \int_{D^h} \Pi_1(C^*_1, C_{-1})\ \mathrm{d}\lambda^*_{-1}(C_{-1})\right) \\
&\quad \geq \sum_{h=1}^{h^*} \Pi_1(C^*_1, C^{*h}_{-1})\mu_{-1}(D^h) \\
&\qquad + \left(F_1(\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_m) - 2\varepsilon - \sum_{h=1}^{h^*} \int_{D^h} \Pi_1(C^*_1, C^{*h}_{-1})\,\mathrm{d}\lambda^*_{-1}(C_{-1})\right) \\
&\quad = \sum_{h=1}^{h^*} \Pi_1(C^*_1, C^{*h}_{-1})\mu_{-1}(D^h) \\
&\qquad + \left(F_1(\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_m) - 2\varepsilon - \sum_{h=1}^{h^*} \Pi_1(C^*_1, C^{*h}_{-1})\lambda^*_{-1}(D^h)\right) \\
&\quad \geq F_1(\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_m) - 2\varepsilon - h^*|\Pi_1|_\infty \mathrm{max}_h |\mu_{-1}(D^h) - \lambda^*_{-1}(D^h)|.
\end{aligned}$$

Since the boundary of D^h has measure zero for all h, for any $\varepsilon > 0$ we can choose a $\gamma_\varepsilon > 0$ such that if $\mu_{-1} \in B_{\gamma_\varepsilon}(\lambda^*_{-1})$, then

$$|\mu_{-1}(D^h) - \lambda^*_{-1}(D^h)| < \varepsilon,$$

(see Ash (1972), p. 196, Theorem 4.5.1(c)). Thus, we have for all $\mu_{-1} \in B_{\gamma_\varepsilon}(\lambda^*_{-1})$,

$$F_1(\hat{C}_1, \mu_{-1}) = \int_{P_f(K_{-1})} \Pi_1(\hat{C}_1, C'_{-1})\,\mathrm{d}\mu_{-1}(C'_{-1}) \geq F_1(\lambda^*_1, \ldots, \lambda^*_m) - 3\varepsilon.$$

Let $\lambda'_1 \in \Delta(P_f(K_1))$ be such that

$$\lambda'_1(\{\hat{C}_1\}) = 1.$$

Then we have

$$F_1(\lambda'_1, \mu_2, \ldots, \mu_m) \geq F_1(\lambda^*_1, \ldots, \lambda^*_m) - 3\varepsilon, \quad \text{for all } \mu_{-1} \in B_{\gamma_\varepsilon}(\lambda^*_{-1}).$$

References

Aliprantis, C.D., Border, K.C., 1999. Infinite Dimensional Analysis: A Hitchhiker's Guide. Springer, Berlin.

Ash, R.B., 1972. Real Analysis and Probability. Academic Press, NY.

Carlier, G., 2000. Problémes de Calcul des Variations Issus de la Théorie des Contrats. Thèse, Université de Paris IX Dauphine, UFR Mathématiques de la Décision.

Carlier, G., 2001. A general existence result for the principal–agent problem with adverse selection. Journal of Mathematical Economics 35, 129–150.

Guesnerie, R., 1981. On Taxation and Incentives: Further Remarks on the Limits to Redistribution. Bonn Discussion Paper 89.

Guesnerie, R., 1995. A Contribution to the Pure Theory of Taxation. Cambridge University Press, Cambridge.

Hammond, P., 1979. Straightforward incentive compatibility in large economies. Review of Economic Studies 46, 263–282.

Himmelberg, C.J., Parthasarathy, T., VanVleck, F.S., 1976. Optimal plans for dynamic programming problems. Mathematical of Operations Research 1, 390–394.

Martimort, D., Stole, L., 1997. Communication Spaces, Equilibria Sets and the Revelation Principle Under Common Agency. GSB, University of Chicago, Mimeo.

Monteiro, P.K., Page Jr., F.H., 2002. Catalog Games, Better-Reply Security, and Nash Equilibrium. Typescript, EPGE/FGV.

Myerson, R., 1982. Optimal coordination mechanisms in generalized principal–agent problems. Journal of Mathematical Economics 10, 67–81.

Page Jr., F.H., 1992. Mechanism design for general screening problems with moral hazard. Economic Theory 2, 265–281.

Page Jr., F.H., 1999. Competitive Selling Mechanisms: The Delegation Principle and Farsighted Stability. CORE Discussion Paper 2000/21.

Reny, P.J., 1999. On the existence of pure and mixed strategy Nash equilibria in discontinuous games. Econometrica 67, 1029–1056.

Rochet, J.C., 1985. The taxation principle and multi-time Hamilton–Jacobi equations. Journal of Mathematical Economics 14, 113–128.

Tulcea, A.I., 1973. On pointwise convergence, compactness and equicontinuity in the lifting topology I. Z. Wahrscheinlichkeitstheorie verw. Geb. 26, 197–205.