The University of Alabama

Finance 601 | **Spring 2010**
Finance Theory I | **Professor Schlesinger**

Exam 1

Answer all 5 questions below. You must show how you derive your answers to receive full credit on each question. You should assume that consumers/investors are all risk averse unless told otherwise.

1. State whether each statement below is TRUE or FALSE (or neither) and explain your reasoning. If you can answer the question with more information, describe what else you would need to know. All of the credit for this problem is based on your explanations.

(a) A lottery with equal probabilities of the four monetary payoffs {100, 150, 200, 275} is preferred by some risk averters to a lottery with equal probabilities of the two monetary payoffs {125, 225}.

(b) Lottery A has utility payoffs u = 100 or u = 50, each with a 50% probability. This lottery is less preferred by some risk averters, to the sure utility payoff of u=75.

(c) Consider the Fisher Separation Theorem under certainty. If there is an increase in the risk-free rate, it is possible for the "value of the firm" (i.e. the present value of the cash flows) to increase.

(d) Consider the Fisher Separation Theorem under certainty. If there is an increase in the risk-free rate, it is possible for a shareholder of the firm (i.e. an owner of the firm's cash flows) to become better off.

2. Consider two distributions, F and G, with their supports contained in the interval (a,b). Suppose that there exists some t_0 with $F(t) \leq G(t)\ \forall t < t_0$ and that $F(t) \geq G(t)\ \forall t \geq t_0$.

(a) Is it possible for F to dominate G via first-order stochastic dominance? Explain. [If it is impossible, explain why. If it is possible, what other conditions on F and G are required?]

(b) Is it possible for G to be a mean-preserving increase in risk over F ? Explain. [If it is impossible, explain why. If it is possible, what other conditions on F and G are required?]

3. Consider an investor with preferences exhibiting constant absolute risk aversion (CARA), who invests part of her wealth in a risk-free bond with gross return R_f and the rest in a risky asset with random return $\tilde{R}$, where $E\tilde{R} > R_f$. Prove that any increase in her initial wealth will all be invested in the risk-free bond.

4. Consider a complete market with *S* states of nature. Let π_i denote the state price for state *i* divided by the probability of state *i* (i.e. the "state-price density"). A risk-averse investor has an initial wealth of w_0 to invest in state-contingent claims.

(a) Write out the first order conditions for this optimization problem and explain why the optimal level of consumption in state *i* is a function of π_i: $c_i{*} = c(\pi_i)$.

(b) Suppose that the investor has preferences exhibiting constant absolute risk aversion (CARA). Show that an increase in this investor's wealth w_0 will be used to purchase an equal amount of additional consumption in all S states of nature. (Note that this is equivalent to investing all of the extra wealth into a risk-free bond.)

5. A consumer receives an income of *y* at dates *t* = 0 and *t* = 1. The risk-free rate of interest is assumed to be r_f, where $0 < r_f < \delta$. The consumer chooses a level of savings *s* to maximize her lifetime utility:

$$\max_s \; V(s) \equiv u(y-s) + \tfrac{1}{1+\delta} u(y + s(1+r_f)),$$

where the felicity function (i.e. the utility function) *u* is increasing and concave. Define $-u''(c)/u'(c)$ as the measure of absolute felicity at consumption level *c* and assume that it is constant for all *c*, $-u''(c)/u'(c) = K$, where *K* >0.

(a) Will the consumer save or borrow or neither? Explain fully.

(b) What is the effect of an increase in *y* (in both periods) on the optimal level of savings?

(c) Recall that the preferences above also imply that absolute prudence is constant $-u'''(c)/u''(c) = K$ (same *K* as above). [You don't need to show this.] Suppose that the consumer's income at date *t* = 1 is risky, so that she receives a random amount $y + \tilde{\varepsilon}$, where $E\tilde{\varepsilon} = 0$. Will the consumer save more or less than in part (a) – or exactly the same amount as in part (a)? Explain.

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Exam 2

Answer all 5 questions below.

1. Suppose that a one factor APT applies and that securities *A* and *B* have returns specified as follows, where the $\tilde{\varepsilon}_i$ are small, mutually independent noise terms. We assume that $E\tilde{f}_1 = 0$.

$$\tilde{R}_A = 1.00 + 2\,\tilde{f}_1 + \tilde{\varepsilon}_A \qquad \tilde{R}_B = 0.95 + 3\,\tilde{f}_1 + \tilde{\varepsilon}_B .$$

(a) What is the implied risk-free rate R_f and what is the price of $\tilde{f}_1$ risk (call it λ_f)?

(b) Factors such as $\tilde{f}_1$ are not unique. For example, we can define $\tilde{g}_1 = -\tilde{f}_1$ and use $\tilde{g}_1$ as the single factor. Thus, we could have written $\tilde{R}_A = 1.00 + b_A\tilde{g}_1 + \tilde{\varepsilon}_A$, where b_A denotes the asset's sensitivity to factor $\tilde{g}_1$. How does the price of the factor $\tilde{g}_1$ risk (call it λ_g) compare to λ_f that you determined in part (a)? Explain.

2. Consider a dynamic, two-period model of investment. The individual has an initial wealth of $\$w$ and utility *u* belonging to the HARA class, with risk tolerance given by $T_u(c) = a + bc$. After investing $\$w$ in Arrow-Debreu securities at date $t = 0$, the individual realizes an intermediate return of $\$z$ at the end of the period, which she re-invests optimally in Arrow-Debreu securities at the start of date $t = 1$. We assume that there is no intermediate consumption at the beginning of date $t = 1$ and that the risk-free rate and rate of time discounting are both zero ($\delta = r_f = 0$). For every realized first-period return *z*, we define the Value Function $v(z)$ as

$$v(z) \equiv \max_{C_S} \sum_{s=1}^{S} p_s u(c_s(z)) \quad \text{s.t.} \quad z = \sum_{s=1}^{S} p_s \pi_s c_s(z)$$

(a) Derive the first order conditions for determining $v(z)$.

(b) Show that for any arbitrary state of nature *k*, it follows that $c'_k(z) = \dfrac{T_u(c_k(z))}{\sum_{s=1}^{S} p_s \pi_s T_u(c_s(z))}$.

(c) Show that the risk tolerance of *v* is the same as the risk tolerance of *u*, i.e. $T_v(z) = T_u(z)$.

3. Consider a one-period complete Arrow-Debreu exchange economy with *N* individuals and *S* states of nature, with no consumption at the beginning of the period. The aggregate wealth in the economy in state *s* is denoted as $z(s)$. Suppose that $z(1) > z(2) > \ldots > z(S)$. Show that $\pi(1) < \pi(2) < \ldots < \pi(S)$, where $\pi(s)$ denotes the equilibrium state-price density for state *s*.

4. Consider a one-period complete Arrow-Debreu exchange economy with N individuals and S states of nature. Assume that each individual i has an initial wealth of ω_{i0} at the start of the period as well as an endowment of ω_{is} Arrow-Debreu securities for each state s. All individuals are assumed to have CRRA preferences with the same degree of relative risk aversion γ. They also have the same rate of time discounting for delayed consumption, $\delta > 0$. Individual i consumes an amount c_{i0} at the start of the period and invests the rest of his wealth in Arrow-Debreu securities. Define $z_0 \equiv \sum_{i=1}^{N} \omega_{i0}$ as the aggregate wealth at the beginning of the period and define $z(s) \equiv \sum_{i=1}^{N} \omega_{is}$ as the aggregate wealth at the end of the period, if state s occurs. We assume that $z(s)$ is different in every state. Thus we can think of $z(\tilde{s})$ as the random aggregate wealth at the end of the period.

(a) Derive the first order conditions for one of the above individuals.

(b) Suppose that the Arrow-Debreu prices for contingent claims are already the equilibrium prices. Further suppose that $z(\tilde{s}) = z_0(1+\tilde{\varepsilon})$, where $E\tilde{\varepsilon} = 0$. Determine whether the equilibrium risk-free rate r_f is larger or smaller than δ. Explain your answer.

5. Consider a principal-agent model in which the principal owns the contingent claim (x_1, x_2), where $x_1 > x_2$. The probability of state 1 depends upon agent effort e, with $p_1(e)$ a continuous, increasing and concave function. Effort by the agent costs c units of utility per unit of effort. The principal and agent are both risk averse. The level of effort is assumed to be **observable**. The agent is paid a fee that is contingent on the state of nature, which we denote by the pair (a_1, a_2). Since effort is observable, these state-contingent payments are only paid if effort meets or exceeds some pre-specified level, e^*.

(a) Write out the first order conditions for determining the optimal agent's payment (a_1^*, a_2^*) and the required effort level e^*.

(b) Will the contract determined in part (a) entail first-best efficient risk sharing? Explain.

(c) Suppose that a new court ruling determines that the contingent payment (a_1^*, a_2^*) must be made to the agent, and that evidence about effort is not admissible in court. Assuming that the contract pair (a_1^*, a_2^*) does not change, will the agent voluntarily supply the amount of effort e^*, that you determined in part (a)? Explain.