PRICE DISCRIMINATION, "ADJUSTED CONCAVITY," AND OUTPUT CHANGES UNDER CONDITIONS OF CONSTANT ELASTICITY

John P. Formby,
Stephen K. Layson, and
W. James Smith

Working Paper No. 21

Working Paper Series
Department of Economics, Finance
and Legal Studies
College of Commerce and Business
Administration
University of Alabama

Price Discrimination, "Adjusted Concavity," and Output Changes Under Conditions of Constant Elasticity

The economic consequences of third degree price discrimination have been extensively investigated. Beginning with Joan Robinson's classic discussion in The Economics of Imperfect Competition, a focal point has been the analysis of the effects of price discrimination on output. Considerable discussion in the literature has been devoted to the case of linear demand, the most recent example of which is Schmalensee (1981).[1] As has been well known since the early 1900's, in this case, discrimination does not change output. Under conditions of non-linear demand, the effect of third degree price discrimination is much less certain and far more complicated. Despite increasing analytical sophistication and effort, no new conclusions on output changes have been forthcoming since the work of Edwards (1950). In this paper, we investigate an important class of non-linear functions and present a definitive analysis of the output effects of third degree price discrimination under constant elasticity conditions.

In the general non-linear case, the effect of discrimination on output may be either positive or negative.[2] Joan Robinson (1933) developed the local "adjusted concavity" criterion to determine the output effects of discrimination for two sub-markets with non-linear demands. Edwards (1950) raised the possibility that Robinson's criterion could lead to erroneous conclusions when the local assumption was relaxed and proposed sufficient global restrictions on the curvature of the demand functions under which the "adjusted concavity" criterion is correct. Greenhut and Ohta (1976) confirmed Edwards' suspicion about the fallibility of the "adjusted concavity" criterion by presenting a counterexample consisting of two constant elasticity demand schedules.[3] Samuelson (1947), Löfgren (1977), Finn (1974), and Silberberg (1970) employing Lagrangean techniques suggested by Leontief (1940) verify Edwards' global restrictions showing that if the more elastic curve is concave (convex) and the less elastic curve is convex (concave) third degree price discrimination will increase (decrease) output.[4]

The case in which sub-market demand or supply curves possess the same general curvature has proven particularly resistant to analysis as has generalization to n-markets. Schmalensee (1981, p. 245) observes, "If all demand functions are strictly concave or convex..., there is apparently no simply way to tell if monopolistic discrimination will raise or lower total output." Silberberg (1970, p. 86) reaches a similar conclusion. A general class of functions in this category consists of constant elasticity demand and supply functions. Schmalensee (1981, p. 245) cites this case as one apparently not yielding definitive output effects.

In our analysis of constant elasticity demand and supply functions, we employ Lagrangean techniques to examine the effects of discrimination on quantity in input and output markets. For input markets, we prove in very general terms that monopsonistic discrimination always increases employment. The proof is global in nature and extends to n-markets. On the output side, monopolistic discrimination is shown to increase output over a wide range of elasticities for the n-market case. As an interesting corollary, we find that, for the constant elasticity case, Joan Robinson's "adjusted concavity" criterion is always incorrect when predicting directional changes in output markets. We argue that the "adjusted concavity" criterion involves a misuse of differential calculus and is not a valid local criterion despite a vast literature to the contrary.

I. ANALYSIS

Consider a monopolist selling in n distinguishable markets with associated sub-market demand curves of the constant elasticity type, i.e.,

$$q_i = a_i P_i^{-\varepsilon_i} \qquad i = 1, \ldots\ldots n, \qquad (1)$$

where q_i is the quantity demanded in the ith market, P_i is the price charged in that market and ε_i is demand elasticity ($\varepsilon_i > 1$). Following the relevant literature (see Schmalensee (1981) for discussion of this point) q_i is assumed to depend only upon P_i for i = 1, ..., n, and marginal cost M_0, is assumed constant.[5]

Under simple monopoly, profits are maximized by charging all buyers a common price P. The aggregate output of the simple monopolist is given by:

$$q_s = \sum_{i=1}^{n} a_i P^{-\varepsilon_i} \quad . \qquad (2)$$

Profit maximization requires that:

$$MR_s = P\ (1 - 1/\varepsilon) = M_0, \qquad (3)$$

where MR_s is marginal revenue and ε the elasticity (at the profit maximum) corresponding to aggregate output given by (2). It is easily shown that ε is the weighted average of the elasticities of the individual markets as given by:

$$\varepsilon = \sum_{i=1}^{n} w_i \varepsilon_i, \qquad (4)$$

where $w_i = a_i P^{-\varepsilon_i} / \sum_{j=1}^{n} a_j P^{-\varepsilon_j} = q_{is}/q_s$ with q_{is} denoting the quantity demanded in the ith market under simple monopoly. For convenience and without loss of generality, we define the units of q such that P = 1 with concomitant implications that $M_0 = 1 - 1/\varepsilon$, $q_{is} = a_i$ and,

$$\varepsilon = \sum_{i=1}^{n} (a_i / \sum_{j=1}^{n} a_j)\varepsilon_i. \tag{5}$$

Under discrimiatory monopoly, the firm maximizes profit by equating marginal cost and marginal revenue in each market, i.e.,

$$MR_1 = \ldots = MR_n = M_0 \text{ or } P_{1d}(1 - 1/\varepsilon_1) = \ldots = P_{nd}(1 - 1/\varepsilon_n) = M_0, \tag{6}$$

where MR_i denotes marginal revenue in sub-market i and P_{id} is the price charged. The discriminatory monopolist's output, q_d, is given by:

$$q_d = \sum_{i=1}^{n} q_{id} = \sum_{i=1}^{n} a_i[M_0/(1 - 1/\varepsilon_i)]^{-\varepsilon_i}, \tag{7}$$

where q_{id} is the output demanded in the ith market under discrimination.

The question of whether third degree price discrimination increases, decreases, or leaves output unchanged, can be addressed by the employment of Lagrangean techniques. We minimize the difference between q_d and q_s with respect to the sub-market elasticities subject to the constraint implied by (4). The Lagrangean function is given by:

$$L = \sum_{i=1}^{n} a_i[M_0/(1 - 1/\varepsilon_i)]^{-\varepsilon_i} - \sum_{i=1}^{n} a_i + \lambda(\bar{\varepsilon} - \sum_{i=1}^{n} w_i\varepsilon_i) \tag{8}$$

where $\bar{\varepsilon}$, the value of the elasticity of the aggregate demand function at the simple monopoly price, is to be held constant. In short, we investigate the behavior of output as sub-markets elasticities change subject to the important restriction that the changes must leave the elasticity of aggregate demand (at the simple monopoly price) unchanged. The impetus for this procedure is the importance of the relative elasticities to third degree discrimination. As is well known, when sub-market elasticities at the simple monopoly price are equal, discrimination does not take place and output does not change. For this reason, we take the point of equal elasticities as a point of reference and departure in examining the first and second order conditions. This is particularly appropriate because Joan Robinson's "adjusted concavity" criterion and other local criteria require differential changes around equal elasticities in their derivation.

The first order conditions are given by:

$$L_i = q_{id} \{\ln[M_0/(1 - 1/\varepsilon_i)] + 1/(1 - 1/\varepsilon_i)\} - w_i\lambda = 0, \; i = 1, \ldots, n \qquad (9)$$

$$L = \bar{\varepsilon} - \sum_{i=1}^{n} w_i\varepsilon_i = 0.$$

Under conditions of equal elasticities it is easily seen that the first order conditions are satisfied.[6]

The second order conditions are established by noting that the k^{th} - order principle minor of the bordered Hessian, denoted by $|\bar{H}_k|$, is given by:

$$|\bar{H}_k| = - \sum_{i=1}^{k} w_i^2 \, |H_{ii}| \qquad (10)$$

where

$$|H_{ii}| = \begin{vmatrix} L_{11} & 0 & \cdots & & & 0 \\ 0 & L_{22} & & & & \\ \cdot & & \ddots & & & \\ \cdot & & & L_{i-1,i-1} & & \\ \cdot & & & & L_{i+1,i+1} & \\ & & & & & \ddots \\ 0 & \cdot & \cdot & \cdot & & L_{kk} \end{vmatrix} \quad (11)$$

and where L_{jj} is the second partial of (9) with respect to the j^{th} elasticity and is given by:

$$L_{jj} = q_{id}\, \psi_j \quad (12)$$

where,

$$\psi_j \equiv [\ln P_j]^2 + [2/(1-\varepsilon_j)] \ln P_j - 1/[\varepsilon_i(1 - \varepsilon_i)].$$

The sign of $|\bar{H}_k|$ clearly depends on the signs of the diagonal elements, L_{jj}, of (11). If all the diagonal elements of (11) are positive, then $|\bar{H}_k| < 0$ and the second order condition for a local minimum will be satisfied.[7] Because the sign of L_{jj} is the same as the sign of ψ_j, a sufficient condition for $|\bar{H}_k| < 0$ is that ψ_j be positive for all j. Consider the case where $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_n = \bar{\varepsilon}$, for which the first order conditions are satisfied, so that we are examining small changes of elasticities around the local point for which they are all equal. In this case, $\psi_j > 0$ for all j, as is easily established by noting that $\ln P = \ln[M_0/(1 - 1/\varepsilon_j)] = 0$ and $\varepsilon_j > 1$. As a result, all of the relevant principal minors, $|\bar{H}_k|$, are negative establishing a local minimum at $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_n$. For small divergencies of elasticities from equality, third-degree-price discrimination implies an increase in quantity sold.

We are able to generalize this result beyond differential changes in ε_j by examining the behavior of ψ_j over a wider domain. To establish that $q_d > q_s$ when the elasticities are not all equal, it is sufficient to show that the Lagrangean function given in (8) is convex. The Lagrangean is convex if all the principal minors, $|\bar{H}_k|$, given by (11), are negative. The requirement that $|\bar{H}_k| < 0$ for all k is satisfied if the least elastic market satisfies:[8]

$$\ln P_i = \ln [M_0/(1 - 1/\varepsilon_i)] < [-1/(1 - \varepsilon_i)] [1 - \sqrt{1/\varepsilon_i}]. \qquad (13)$$

The restrictiveness of (13) depends upon the level of M_0 relative to ε_i. The greater the value of ε_i, the greater the value of M_0 permitted. For example, if $\varepsilon_i = 10$, M_0 must be less than .971 (remembering P = 1, this is not a very stringent restriction). If $\varepsilon_i = 1.5$, $M_0 < .481$. For the two sub-market case, extensive search beyond the bounds of (ε_i, M_0) for which (13) holds, failed to produce any example in which third degree price discrimination decreases output in the constant elasticity case.

The above analytics lend themselves to examination of the input market case. The results turn out to be very general. It is only necessary to redefine M_0 as marginal revenue product, P_i as input prices, and note that $q_i = a_i P_i^{\varepsilon_i}$ where $\varepsilon_i > 0$. As in the case of the output market, a constant M_0 is not a restrictive assumption. In the input market case, it is easily seen that ψ_j in (12) is always positive and all the principal minors $|\bar{H}_k|$ are negative. This establishes that $q_d - q_s$ reaches a global minimum of 0 at $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_n$. For all other values of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, $q_d > q_s$.

The standard criterion for determining output changes since 1933 has been that of "adjusted concavity". It is appropriate to contrast its predictions with the results proven above. In the input market case (see Ekelund, Higgins and Smithson (1981, p. 699) for a recent discussion), the "adjusted concavity" criterion states that employment under discrimination will be greater, equal to, or less than, the simple monopsony level of employment as the "adjusted concavity" of the more elastic supply curve is less than, equal to, or greater than, that of the less elastic supply curve. "Adjusted concavity" (denoted by θ_i) is given by $\theta_i = P(-1 + 1/\varepsilon_i)$ where P is the simple monopsony input price. The more elastic supply function always possesses the lesser "adjusted concavity". The criterion predicts that discrimination will always increase employment, and the criterion is always correct.

In the output market case, the "adjusted concavity" criterion states that output under discrimination will be greater, equal to, or less than the simple monopoly output, as the "adjusted concavity" of the demand curve in the relatively elastic market is greater, equal to, or less than, that of the market with the relatively inelastic demand. In the constant elasticity output market case, "adjusted concavity" simplifies to $\theta_i = [1 + 1/\varepsilon_i]$. As a result, the more elastic demand function possesses the lesser "adjusted concavity." The "adjusted concavity" criterion predicts that discrimination will always decrease output. From the results proven earlier, it follows that the "adjusted concavity" criterion is always incorrect in the constant elasticity output market case.

This result is an important one and establishes that the "adjusted concavity" criterion is not a valid local criterion as has been maintained since it was first introduced by Joan Robinson in 1933. This conclusion is at first surprising and deserves further discussion. Two points are relevant. First, as Edwards (1950) and Smith and Formby (1981) establish, given two convex demand curves, as in our analysis, the "adjusted concavity" criterion underestimates the increase in the more elastic market and over-estimates the decrease in the less elastic market. The errors incurred in approximation constitute a systematic bias in the opposite direction to the criterion's prediction and can be large enough to invalidate the criterion's prediction as Greenhut and Ohta (1976) show by example. The argument has been made and accepted in the literature that this will not occur if the difference in the sub-market elasticities (or equivalently if the change in prices in the sub-markets) is small. The second point which we make is relevant to this latter argument. Our analysis shows that this conclusion is incorrect. The reason is that the "adjusted concavity" criterion involves a misuse of differential calculus.

In differential calculus, it is well known that if the change in the independent variable is small, the error incurred will be of second-order of smallness. In Robinson's "adjusted concavity" criterion, when the changes in prices are small the error in approximating the output change in an individual sub-market is indeed small. However, it is not the change in the individual sub-markets on which the prediction of the "adjusted concavity" criterion rests. It is the <u>difference</u> in the predicted changes in those markets. And when the difference of the output increase in one market and the decrease in the other market is taken, the difference is of a second-order of smallness and of the same order as the errors. The errors may be greater than the predicted net change in outputs. In this case, the possibility arises that Robinson's criterion may be incorrect even though the changes in the independent variables are "small". The analysis in this paper conclusively demonstrates that for the output market, the errors are always larger and in the direction which will always invalidate the criterion's prediction. Although this follows from the formal analysis above, an example will more clearly illustrate this point.

Consider the two sub-market output cases where $q_{1d} = P_1^{-5.05}$, $q_{2d} = P_2^{-4.95}$ and $M_0 = .8$. Under simple monopoly, profit maximization implies P = 1 and $q_{1s} = q_{2s} = 1$. The price elasticity of aggregate demand at P = 1 is -5. The "adjusted concavity" criterion utilizes tangents to the sub-market marginal revenue curves at q_{1s} and q_{2s} to predict the changes in output in each of the sub-markets under price discrimination. In this example the predicted increase in market 1 is .0124692 and the predicted decrease in market 2 is -.0125318. The actual changes in market 1 and 2 are .0125631 and -.0124378 with errors of .0000939472 and .0000938829 respectively. Each of the errors is of the same order of magnitude as and of opposite sign to the net predicted change of -.0000626. Either error is sufficient to overturn the criterion's prediction of a decrease in net output. The actual net change in output is a positive .000125289. Output increases rather than decreases. One can examine cases in which the elasticities are even closer in magnitude but the conclusion does not change. The errors in each sub-market become smaller but the predicted net change becomes smaller by the same order of magnitude with the result that the errors always reverse the criterion's prediction.

The technique employed in this paper in contrast to the "adjusted concavity" criterion is a valid use of differential calculus for we consider $q_d - q_s$ as a function of the sub-market elasticities. In employing differentiation we do so at one point rather than two. In this case, the error is of a second-order of smallness.

II. CONCLUSION

This paper extends the analysis of the effects of price discrimination on output and employment to the important class of constant elasticity demand and supply functions. Under these conditions we prove that profit maximizing monopsonsitic price discrimination always increases employment. The proof is global and extends to the n-market case. For the output market we prove that profit maximizing monopolistic price discrimination increases output for a wide range of elasticities. In the output case, we show that Joan Robinson's "adjusted concavity" criterion always leads to incorrect predictions of directional output changes. This proof demonstrates that the "adjusted concavity" criterion is not a valid "local" criterion.

Footnotes

[1]Schmalensee generalizes Robinson's result to n markets and shows that output must increase for welfare to increase.

[2]Under conditions of linear demand, Battalio and Ekelund (1971) show that if one market is not served under simple monopoly, discrimination can increase output. In this paper, we assume both markets are served under each regime.

[3]Greenhut and Ohta (1972, 1975, 1979) prove that spatial discrimination always increases output. We do not take up the spatial question.

[4]Smith and Formby (1981) recently introduced the DMR-SMR criterion, based upon the divergence of the marginal revenues facing the simple (SMR) and discriminating monopolist (DMR), to provide for a globally correct method of determining directional output changes. While the DMR-SMR criterion provides a correct method for determining output changes, it has not lent itself to identification of the underlying market demand functions for which DMR $\gtrless$ SMR. Eklund, Higgins, and Smithson (1981) extend the application of the "adjusted concavity" criterion to the input market. In a recently published paper, Formby, Layson, and Smith (1982) show that the "adjusted concavity" criterion fares no better on the input side by presenting a counterexample to its validity consisting of two quadratic convex supply functions. They also propose an alternative globally correct criterion similar to the DMR-SMR criterion on the output side.

[5]The constancy of M_0 is not restrictive. As Joan Robinson (1933) argued and Silberberg (1970) proves, the slope of M_0 effects only the magnitude of the change in output but not its direction.

[6]For $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_n = \bar{\varepsilon}$ the first order conditions simplify to

$$L_i = a_i[(1/(1 - \bar{\varepsilon}) - \lambda / \sum_{j=1}^{n} a_j] = 0, \quad i = 1, \ldots, n.$$

For $\lambda = \sum_{j=1}^{n} a_j[(1/(1 - \bar{\varepsilon})]$, all the first order conditions are satisfied.

[7]See Chiang (1974, pp. 338-90) for the second-order condition for constrained minimization problems.

[8]The term ψ_i is quadratic and convex in ln P_i. The roots of ψ_i are ln $P_i = [-1/(1 - \varepsilon_i)] [1 \pm \sqrt{1/\varepsilon_i}]$. Therefore if ln P_i is less than the smaller root, $[-1/(1 - \varepsilon_i)][1 - \sqrt{1/\varepsilon_i}]$, ψ_i is positive, and since ln P_i = ln $[M_0/(1 - 1/\varepsilon_i)] < -1/(1 - \varepsilon_i)$, this exhausts the set of feasible ln P_j for which ψ_j is positive. With this restriction on the domain, the sufficient condition for ψ_i to be positive is

$$\ln P_i < [-1/(1 - \varepsilon_i)] [1 - \sqrt{1/\varepsilon_i}], \qquad (14)$$

where market i is the least elastic market. If the least elastic market satisfies (14), then, the other markets must also satisfy (14). Consequently, satisfaction of (14) guarantees that $\psi_j > 0$ for all j and that the Langrangean function is convex.

REFERENCES

Battalio, R.C. and Ekelund, R.B. (1972). "Output Change Under Third Degree Discrimination." Southern Economic Journal, Vol. 39, pp. 285-290.

Chiang, A.C. (1974). Fundamental Methods of Mathematical Economics, 2nd ed. New York: McGraw-Hill.

Edwards, E.O. (1950). "The Analysis of Output Under Discrimination." Econometrica, Vol. 1, pp. 161-172.

Ekelund, R.B., Higgins, R.S., and Smithson, C.W. (1981). "Can Discrimination Increase Employment: A Neoclassical Perspective." Southern Economic Journal, Vol. 47, pp. 664-673.

Finn, T.J. (1974). "The Quantity of Output in Simple Monopoly and Discriminating Monopoly." Southern Economic Journal, Vol. 41, pp. 239-243.

Formby, J.P., Layson S.K., and Smith W.J. (1982). "Discriminatory Changes in Employment." Southern Economic Journal, Vol. 49, pp. 550-555.

Greenhut, M.L. and Ohta H. (1972). "Output Under Alternative Spatial Pricing Techniques." American Economic Review, Vol. 62, pp. 705-713.

_______ (1975). Theory of Spatial Pricing and Market Areas. Durham: Duke University Press.

_______ (1976). "Joan Robinson's Criterion for Deciding Whether Market Discrimination Reduces Output." Economic Journal, Vol. 86, pp. 96-96.

_______ (1979). "Output Effect of Spatial Discrimination Under Conditions of Monopoly and Competition." Southern Economic Journal, Vol. 46, pp. 71-84.

Leontief, W.W. (1940). "The Theory of Limited and Unlimited Discrimination." Quarterly Journal of Economics, Vol. 54, pp. 490-501.

Löfgren, K.G. (1977). "A Note on Output Under Discrimination." Revista Internationale di Science Economiche, Vol. 24, pp. 776-782.

Robinson, J. (1933). The Economics of Imperfect Competition. London: Macmillan and Company.

Samuelson, P.A. (1947). Foundations of Economic Analysis. Cambridge: Harvard University Press.

Schmalensee, R. (1981). "Output and Welfare Implications of Monopolistic Third-Degree Price Discrimination." American Economic Review, Vol. 71, pp. 242-247.

Silberberg, E. (1970). "Output Under Discriminating Monopoly." Southern Economic Journal, Vol. 37, pp. 84-87.

Smith, W.J. and Formby, J.P. (1981). "Output Changes Under Third Degree Price Discrimination: A Reexamination." Southern Economic Journal, Vol. 48, pp. 164-171.